Proceedings of ASME 2010 International Design Engineering Technical Conferences & (IDETC) and Computers and Information in Engineering Conference (CIE)
IDETC/CIE 2010
August 15-18, 2010, Montreal, Canada

DETC2010-29001

SUPPRESSION OF STRING VIBRATION USING A CONSTRAINT ACTUATOR AT ONE BOUNDARY

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ABSTRACT

The problem of a vibrating string subjected to a sudden constraint at one boundary is investigated in this paper. The constraint is imposed by a sleeve that axially moves along the mean position of the string with a small distance. The constraint is applied instantaneously such that the geometry of the string outside the sleeve, immediately after application of the constraint, remains unchanged whereas the length of string covered by the sleeve remains at rest. The change in energy of the string after application and removal of the sleeve is investigated for different values of sleeve travel distance and time of application of the constraint. Analytical and numerical simulation results are first provided for the string vibrating in the first mode, and then for a more general case where the string has arbitrary initial conditions. The results show that the energy content can decrease or increase depending on the time of application of the constraint and sleeve travel distance. This provides the opportunity for active control of string vibration through direct physical interaction with a small portion of the string by using the sleeve as an actuator.

INTRODUCTION

The dynamics of vibrating strings has been a subject of study for very long time but the problem of control of vibrating strings has appeared in the literature only a few decades back. Early work by Dutt and Ramakrishna [1] investigated application of distributed control force to minimize the energy over a part of the string while maintaining the energy at a desired level over the uncontrolled part of the string. Later, Dutt and Ramakrishna [2] extended their work to investigate optimal positions and magnitudes of the applied controlling forces. A different method of vibration suppression was implemented by Myshkis [3], who considered energy dissipation through application of impulsive forces. Hebrard and Henrott [4] investigated energy decay in a string subjected to a constant damping coefficient over an interval of the string length, and proved that the energy decays exponentially in time.

Transverse boundary control problems have been studied by many researchers. Do, K. D. and Pan, J. [6] investigated boundary control of the Transverse motion of motion of flexible marine risers driven by a hydraulic system in presence of disturbances induced by waves, wind, and ocean currents. The controller was designed by using Lyapunovs direct method and the backstepping technique. Bazezew, A. et al. [7] used investigated optimal boundary control of a vibrating beam using of finite element recurrence scheme. Rastgoftar, H. et al. [8] presented a solution to the boundary stabilization problem of the transverse vibration of a composite laminated plate. The control force is based on feedback velocities at the boundaries of the plate. Several researchers have studied boundary control of travelling strings and other continuous systems. Shahruz and Kurmaji [5] studied the control of axially moving strings by applying vertical control forces at one of the supports. Fung, et al. [9] proposed the use of a mass-spring-damper type feedback controller at one boundary for energy dissipation of an axially moving string and verified exponential stability of the system. Chen and Zhang [10] investigated stabilization of axially moving strings by using tensioners that act as actuators. The asymptotic stability of the equilibrium configuration of the string in the presence of perturbations and external disturbances was established using Lyapunov stability theory. Li, et al. [11] used boundary control to suppress vibration of an axially moving string based on a generalized nonlinear model and by using Hamilton's Principle. Qu [12] implemented iterative learning control technique on a system comprised of a stretched string and a transporter where the transporter acts as a vibrator when it is subjected to external disturbances.

In this paper we investigate the dynamics of a string subjected to a sudden constraint at one of its boundary applied by a sleeve. A formal problem statement and a list of the assumptions made in our analysis is provided in section 2. In section 3 we use analytical methods to derive the geometry of motion and total energy of the string after application of the constraint. In section 4 we investigate the motion of the string after removal of the constraint. In section 5 we depict the behavior of the string and provide simulation results for change in energy due to application and removal of the constraint. Section 6 provides concluding remarks.

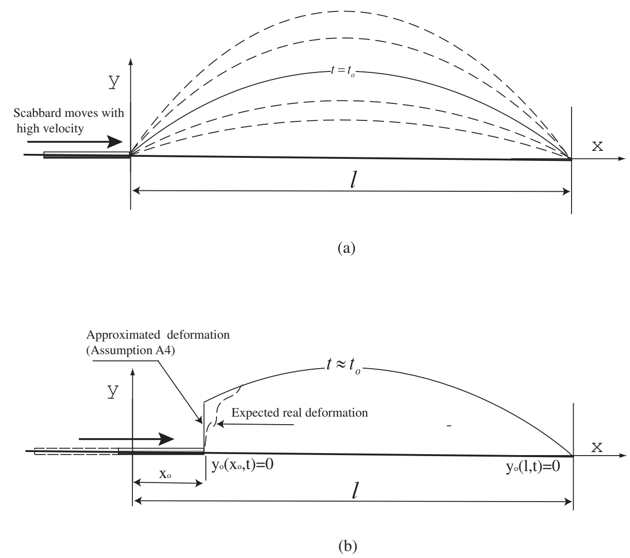

Figure 1. A STRING VIBRATING AND A CONSTRAINT (SLEEVE) IS APPLIED: (a) A STRING VIBRATING AT ARBITRARY $t \approx t_o$ CONSTRAINT IS APPLIED AT HIGH VELOCITY v_o , (b) THE CONSTRAINT IS COMPLETELY APPLIED AND THE STRING AT $t > t_o$ WILL VIBRATE ACCORDING TO THE INITIAL AND BOUNDARY CONDITIONS AT $t \approx t_o$.

PROBLEM STATEMENT AND ASSUMPTIONS

Consider a string initially vibrating in the fundamental mode, as shown in Fig.1(a). Assume a constraint to be applied at one boundary of the string by moving a sleeve at some arbitrary time $t = t_o$. The sleeve is assumed to move along the mean position of the string for a distance $x = x_o$ with high velocity such that the constraint is instantaneously applied at $t = t_o$. We investigate the effect of applying and removing the constraint on the total energy of the string under the following assumptions:

A1. The string is homogeneous and has a constant mass per unit length denoted by ρ . The tension in the string is equal to T and remains constant at all times. The string undergoes transverse vibration in the xy plane and is not affected by gravity. The string has no bending energy due to small cross section.

A2. The string is initially vibrating in the fundamental mode as shown in Fig.1(a).

A3. The applied constraint is represented by a smooth sleeve that moves along the equilibrium position of the string, as shown in Fig.1(b). The part of the string inside the sleeve remains at rest as long as constraint is applied.

A4. The velocity of the sleeve during application of the constraint is much higher than string velocity such that the configuration of the string outside the sleeve after application of the constraint is the same as that before application of the constraint. This is shown in Fig.1(b).

A5. The amplitude of oscillation of the string is small and therefore the equation of motion of the string can be expressed by the standard relation [13]

$$\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{1}{c^2}\left(\frac{\partial^2 y}{\partial t^2}\right), \qquad c \triangleq \sqrt{T/\rho} \tag{1}$$

where $y(x,t)$ is the displacement of the string at a distance x from the origin at time t.

A6. The string has no internal damping, i.e., the energy of the string will remain conserved during free vibration.

A7. The displacement of the sleeve x_o is small and therefore $y_o(x_o,t)$ is small for any t when the constraint is applied. Fig.1(b) shows an exaggerated representation of the discontinuity when the string is applied. In reality, however, the string will have a continuous curvature.

EFFECT OF APPLICATION AND REMOVAL OF THE CONSTRAINT

The equation of motion of the string before application of the constraint is shown in Fig.(1a) and is described by the relation

$$y_o(x,t) = A_o \sin\lambda_o x \sin\omega_o t \tag{2}$$

where $\lambda_o = \frac{\pi}{l}$, $\omega_o = c\lambda_o$ and $c \triangleq \sqrt{T/\rho}$. After application of the constraint, the string motion can be obtained by solving Eq.(1) and applying appropriate boundary and initial conditions. A general solution to Eq.(1) can be written as follows [13]

$$y(x,t) = (A\sin\lambda x + B\cos\lambda x)\,(C\sin\omega t + D\cos\omega t) \tag{3}$$

where A, B, C, D are constants, ω is the circular frequency and λ is related to ω by the relation

$$\omega \triangleq c\lambda \tag{4}$$

Equation of motion after application of the constraint

The motion of the string immediately after application of the constraint can be obtained from Eq.(3) as follows

1. $y(x_o,t) = 0$, implies

$$A = -B.\tan\lambda x_o \tag{5}$$

2. $y(l,t) = 0$ implies

$$A = -B.\tan\lambda l \tag{6}$$

From Eqs.(5) and (6), we get
where $n = 1,2,\cdots$. Substituting the expression for λ_n in Eq.(3) we obtain the general solution as follows

$$y(x,t) = \begin{cases} 0 & : x \in [0,x_o] \\ \sum \sin\lambda_n(l-x)(C_n\sin\omega_n t + D_n\cos\omega_n t) & : x \in [x_o,l] \end{cases} \tag{7}$$

where $\omega_n = c\lambda_n$. The constants C_n and D_n in Eq.(7) can be obtained from the initial conditions. If the string is initially vibrating in the first mode and the constraint is applied at some arbitrary time $t = t_o$, the initial conditions can be written as

1. $y(x,0) = y_o(x,t_o) = A_o\sin\lambda_o x\sin\omega_o t_o \quad : x \in [x_o,l]$
   Substituting $t = 0$ in Eq.(7) gives

$$A_o.\sin\lambda_o x.\sin\omega_o t_o = \sum D_n \sin\frac{n\pi(l-x)}{l-x_o} \tag{8}$$

By multiplying both sides with $\sin\frac{n\pi(l-x)}{l-x_o}$ and integrating from x_o to l we get

$$\begin{aligned} D_n &= \frac{2.A_o\sin\omega_o t_o}{l-x_o}\int_{x_o}^{l}\sin\frac{\pi}{l}x.\sin\frac{n\pi(l-x)}{l-x_o}dx \\ &= \frac{2.A_o\sin\omega_o t_o}{l-x_o}\frac{(-1)^n\lambda_n}{\lambda_o^2-\lambda_n^2}\sin\frac{x_o}{l}\pi \end{aligned} \tag{9}$$

where $\lambda_n = \frac{n\pi}{l-x_o}$ and $\lambda_o = \frac{\pi}{l}$

2. $\frac{d}{dt}y(x,0) = \frac{d}{dt}y_o(x,t_o) = A_o\omega_o\sin\lambda_o x\cos\omega_o t_o \quad : x \in [x_o,l]$
   Differentiating Eq.(7) with respect to t and substituting $t = 0$, we get

$$A_o\omega_o\sin\lambda_o x\cos\omega_o t_o = \sum C_n\frac{nc\pi}{l-x_o}\sin\frac{n\pi(l-x)}{l-x_o} \tag{10}$$

By multiplying both sides with $\sin\frac{n\pi(l-x)}{l-x_o}$ and integrating from x_o to l we get
$C_n = \frac{2.A_o\omega_o\cos\omega_o t_o}{nc\pi}\int_{x_o}^{l}\sin\frac{\pi}{l}x.\sin\frac{n\pi(l-x)}{l-x_o}dx$ Recall the result of the integration from Eq(9), then C_n becomes

$$\begin{aligned} C_n &= \frac{2.A_o\omega_o\cos\omega_o t_o}{nc\pi}\int_{x_o}^{l}\sin\frac{\pi}{l}x.\sin\frac{n\pi(l-x)}{l-x_o}dx \\ &= \frac{2.A_o\omega_o\cos\omega_o t_o}{nc\pi}\frac{(-1)^n\lambda_n}{\lambda_o^2-\lambda_n^2}\sin\frac{x_o}{l}\pi \end{aligned} \tag{11}$$

The coefficients C_n and D_n depend on x_o and t_o and this implies that the motion described by Eq.(7)) can be controlled by changing x_o and/or t_o.

Energetics of constraint application

In this section we investigate the effect of application of the constraint on the total energy of the string. The total energy of the string before application of the constraint is the energy of the string vibrating freely in the first mode, which is given as [14]

$$E_o = \frac{T}{2}\int_0^l\left(\frac{\partial y_o(x,t)}{\partial x}\right)^2 dx + \frac{\rho}{2}\int_0^l\left(\frac{\partial y_o(x,t)}{\partial t}\right)^2 dx \tag{12}$$

By substituting the expression for $y_0(x,t)$ from Eq.(2) into Eq.(12), we get

$$E_o = \frac{T}{2}\frac{l}{2}A_o^2.\lambda_o^2.\sin^2\omega_o t + \frac{\rho}{2}\frac{l}{2}A_o^2.\omega_o^2.\cos^2\omega_o t$$

Recall and substitute $\lambda_o = \frac{\pi}{l}$, $\omega_o = c\lambda_o = \frac{c\pi}{l}$ and $T = \rho.c^2$ in the equation above. The total energy of the string before application of the constraint then takes the compact form

$$E_o = \frac{T.A_o^2\pi^2}{4l} \tag{13}$$

After application of the constraint, the motion of the string is given by Eq.(7), with C_n and D_n given by the expressions in Eqs.(9) and (11). This sudden change in boundary conditions will change (increase or decrease) the total energy of the string. The total energy of the string after application of the constraint can be obtained from the following equation

$$E_n = \frac{T}{2}\int_{x_o}^{l}\left(\frac{\partial y(x,t)}{\partial x}\right)^2 dx + \frac{\rho}{2}\int_{x_o}^{l}\left(\frac{\partial y(x,t)}{\partial t}\right)^2 dx \tag{14}$$

where $\left(\frac{\partial y(x,t)}{\partial t}\right)^2 = \left(\sum \omega_n.\sin\lambda_n(l-x).(C_n\cos\omega_n t - D_n\sin\omega_n t)\right)^2$
and $\left(\frac{\partial y(x,t)}{\partial x}\right)^2 = \left(\sum -\lambda_n.\cos\lambda_n(l-x).(C_n\sin\omega_n t + D_n\cos\omega_n t)\right)^2$
By substituting the above expressions and simplifying, we get

$$\begin{aligned} E_n &= \frac{T}{2}\int_{x_o}^{l}\left(\sum -\lambda_n.\cos\lambda_n(l-x).f_{1n}(t)\right)^2 dx \\ &+\frac{\rho}{2}\int_{x_o}^{l}\left(\sum \omega_n.\sin\lambda_n(l-x).f_{2n}(t)\right)^2 dx \\ &= \frac{T\pi^2}{4(l-x_o)}\sum n^2(C_n^2+D_n^2) \end{aligned} \tag{15}$$

where $f_{1n}(t) = C_n\sin\omega_n t + D_n\cos\omega_n t$, and
$f_{2n}(t) = C_n\cos\omega_n t - D_n\sin\omega_n t$

It is important to note that E_n is a function of x_o,C_n,T and D_n, of which C_n and D_n are functions of T,x_o,A_o and t_o (as shown in Eqs.(9) and (11)) and T&A_o are constants. As a result, $E_n = E_n(t_o, x_o)$. This result is important because it will give us information about the change in total energy before and after applying the constraint($E_n - E_o$). If we define the increase in total energy as the *work done* by the constraint on the string, the work done will be given by the expression

$$W_d = E_n - E_o \tag{16}$$

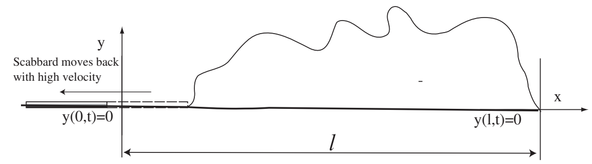

Figure 2. THE CONSTRAINT (SLEEVE) ON THE VIBRATING STRING IS REMOVED AT t_r.

Substituting Eqs.(13) & (15) in Eq.(16), we get

$$W_d = \frac{T\pi^2}{4(l-x_o)}\sum n^2(C_n^2+D_n^2) - \frac{T.A_o^2\pi^2}{4l} \tag{17}$$

Substituting the values of C_n and & D_n from Eqs.(9) and (11))in the equation above gives

$$\begin{aligned} W_d = \frac{T\pi^2A_o^2}{(l-x_o)}\sum n^2\frac{\lambda_n^2}{(\lambda_o^2-\lambda_n^2)^2}\sin^2\frac{x_o}{l}\pi\times \\ \left(\frac{\omega_o^2}{(c.n.\pi)^2}\cos^2\omega_o t_o + \frac{1}{(l-x_o)^2}\sin^2\omega_o t_o\right) - \frac{T.A_o^2\pi^2}{4l} \end{aligned} \tag{18}$$

It is clear from Eq.(18) that the work done depends on x_o and t_o. In other word, changing of x_o and/or t_o may cause negative or positive work to be done. Since negative work done will reduce the total energy of the string, a feedback control strategy can be potentially developed for suppressing the vibration of the string. This is the motivation behind the paper.

Equation of motion after removal of the constraint

We analyse the dynamics of the string after removal of the constraint at time $t = t_r$. We begin with the general solution for the motion of a vibrating string which is given by the relation

$$y_r(x,t) = (\gamma_1\sin\lambda x + \gamma_2\cos\lambda x)\,(\alpha\sin\omega t + \beta\cos\omega t) \tag{19}$$

Immediately after removal of the constraint, the boundary conditions can be used to describe the string geomtery as follows

1. $y_r(0,t) = 0$ implies $\gamma_2 = 0$
2. $y_r(l,t) = 0$ implies $\sin\lambda l = 0$ which in turn implies

$$\lambda = \lambda_k = \frac{k\pi}{l} \tag{20}$$

where $k = 1, 2, \cdots$

Substitution of the expression for λ_k from Eq.(20)and $\gamma_2 = 0$ in Eq.(19 yields

$$y_r(x,t) = \sum_{k=0}^{\infty} \sin\frac{k\pi}{l}x(\alpha_k \sin\frac{kc\pi}{l}t + \beta_k \cos\frac{kc\pi}{l}t) \quad (21)$$

Before removal of the constraint, the string vibration is described by Eq.(7). Therefore, the geometry of the string at the moment when the constraint is removed can be written as

1. $y_r(x,0) = y(x,t_r)$
   Substituting $t = 0$ in Eq.(21) results in

$$\sum \sin\frac{n\pi(l-x)}{l-x_o}(C_n \sin\frac{nc\pi}{l-x_o}t_r + D_n \cos\frac{nc\pi}{l-x_o}t_r) = \sum \beta_k \sin\frac{k\pi}{l}x \quad (22)$$

Multiplying both sides with $\sin\frac{k\pi x}{l}$ and integrating from 0 to l, we get

$$\beta_k = \frac{2.f_r(t_r)}{l}\int_{x_o}^{l} \sin\frac{k\pi}{l}x.\sin\frac{n\pi(l-x)}{l-x_o}dx = \frac{2.f_r(t_r)}{l}\sum_{n=1}^{\infty}\frac{(-1)_n^{\lambda}}{\lambda_k^2-\lambda_n^2}\sin\frac{k.\pi}{l}x_o \quad (23)$$

where $f_{rn}(t_r) = C_n \sin\frac{nc\pi}{l-x_o}t_r + D_n \cos\frac{nc\pi}{l-x_o}t_r$, $\lambda_n = \frac{n\pi}{l-x_o}$, and $\lambda_k = \frac{k\pi}{l}$.

2. $\frac{d}{dt}y_r(x,0) = \frac{d}{dt}y(x,t_r)$ Differentiating Eq.(22) with respect to t and substituting $t = 0$ we get

$$\sum \frac{nc\pi}{l-x_o}\sin\frac{n\pi(l-x)}{l-x_o}(C_n \cos\frac{nc\pi}{l-x_o}t_r - D_n \sin\frac{nc\pi}{l-x_o}t_r) = \sum \frac{ck\pi}{l}\alpha_k \sin\frac{k\pi}{l}x \quad (24)$$

Multiplying both sides with $\sin\frac{k\pi x}{l}$ and integrating from 0 to l, we get

$$\alpha_k = \frac{2\dot{f}_{rn}(t_r)}{ck\pi}\int_{x_o}^{l}\sin\frac{k\pi}{l}x\sin\frac{n\pi(l-x)}{l-x_o}dx = \sum_{n=1}^{\infty}\dot{f}_{rn}(t_r)\frac{(-1)^n\lambda_n}{\lambda_k^2-\lambda_n^2}\sin\frac{k.\pi}{l}x_o \quad (25)$$

where $\dot{f}_{rn}(t_r) = \frac{nc\pi}{l-x_o}(C_n \cos\frac{nc\pi}{l-x_o}t_r - D_n \sin\frac{nc\pi}{l-x_o}t_r)$ The energetics of constraint removal is not discussed here but it can be shown that there is no gain or loss of energy when the constraint is removed.

NUMERICAL SIMULATIONS

Consider a string with the following specifications: $T = 1N$, $\rho = 0.025\ kg/m$, $l = 4\ m$, and initial motion of the string

$$y_o(x,t) = \sin\frac{\pi}{4}x.\sin\frac{\pi}{2}t$$

The simulation is done for different values of x_o and t_o as described below. Figure 3 shows that negative work is done by constraint application for t_0 approximately satisfying $1.9 < t_o < 2.1$. The maximum energy reduction depends on x_o, but for all values of x_o negative work is done for $t_o = 2$ sec, when the string passes through the mean position. This is not surprising because at the mean position the potential energy is zero and the kinetic energy is maximum. On the contrary, as the string moves away

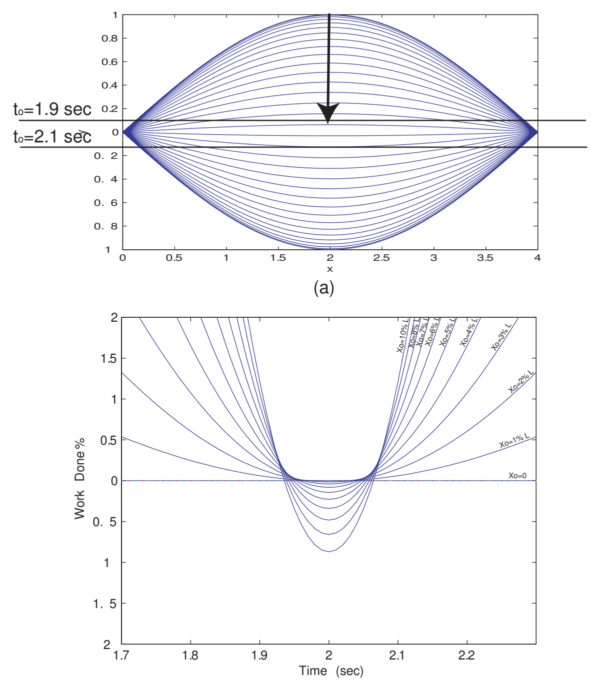

Figure 3. PERCENTAGE DIFFERENCE IN TOTAL ENERGY DUE TO CHANGE IN x_o AND t_o.

from its mean position, the work done becomes positive and increases as the string moves farther from its mean position. This can be explained by the fact that the potential energy of the string is increased due to sudden and large change in the slope of the string when the constraint is applied. Even though the constraint will remove kinetic and potential energy from a small portion of the string, the increase in potential energy due to application of the constraint is much higher. Figures 4 and 5 show the string behavior immediately after application and removal of constraint with $x_o = 0.05l$ and two different values of t_o. It is clear from the figures that the energy of the string is reduced when the string passes through the mean position, and is increased when the string is far away from the mean position.

CONCLUSION

We investigated the dynamics of a vibrating string due to sudden application and removal of a constraint at one of its boundaries. The constraint is applied by moving a sleeve over the string by a short distance, resulting in a sudden change in length of the vibrating string. Under the assumptions of small displacements and linearity, it was shown that application of the constraint can increase or decrease the total energy of the string depending on the time of application of the constraint. When the constraint is removed, the energy of the string remains unchanged. If the constraint is always applied at times when it removes energy from the string, cyclic application and removal of constraint will result in vibration suppression. The analysis in this paper was based on vibration of the string in its fundamental mode at the initial time and development of a vibration control strategy will require that the analysis be extended to a general mode of vibration. If successful, the methodology can be used for vibration control of soft structures such as membranes through physical interaction with a relatively small section of the structure near the boundary.

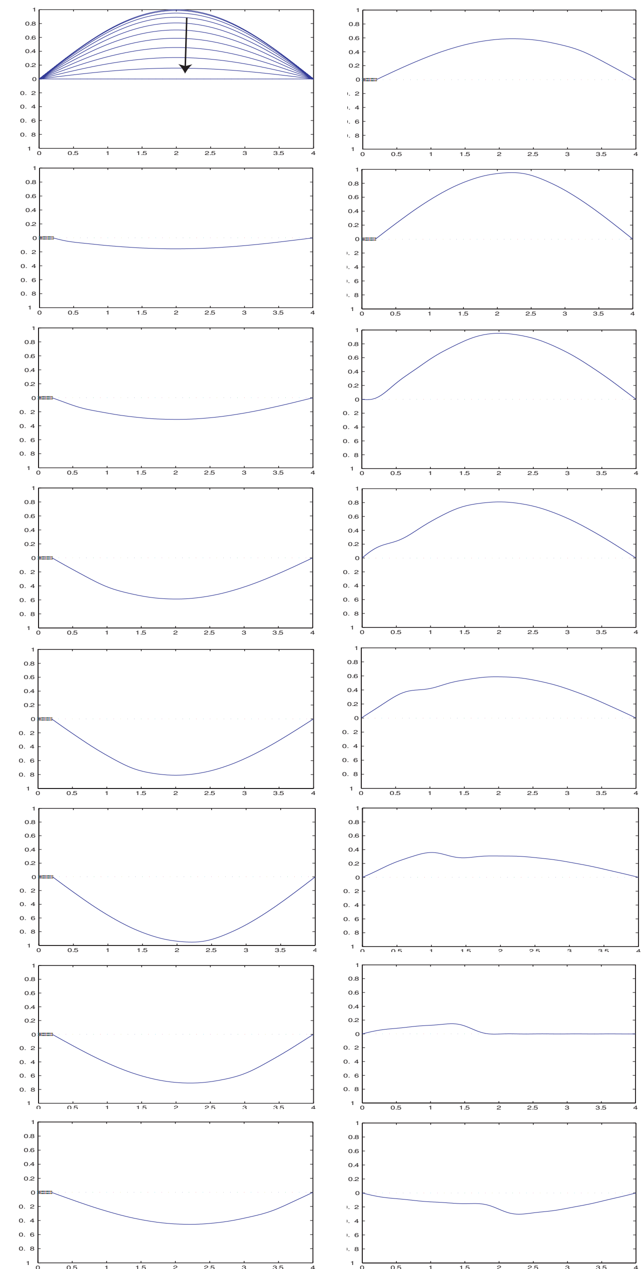

Figure 4. APPLICATION AND REMOVAL OF THE CONSTRAINT WITH $x_o = 0.05l$ AT $t_o = 2$ SEC, THE PERCENTAGE OF THE WORK DONE IS -0.1428 $\%$.

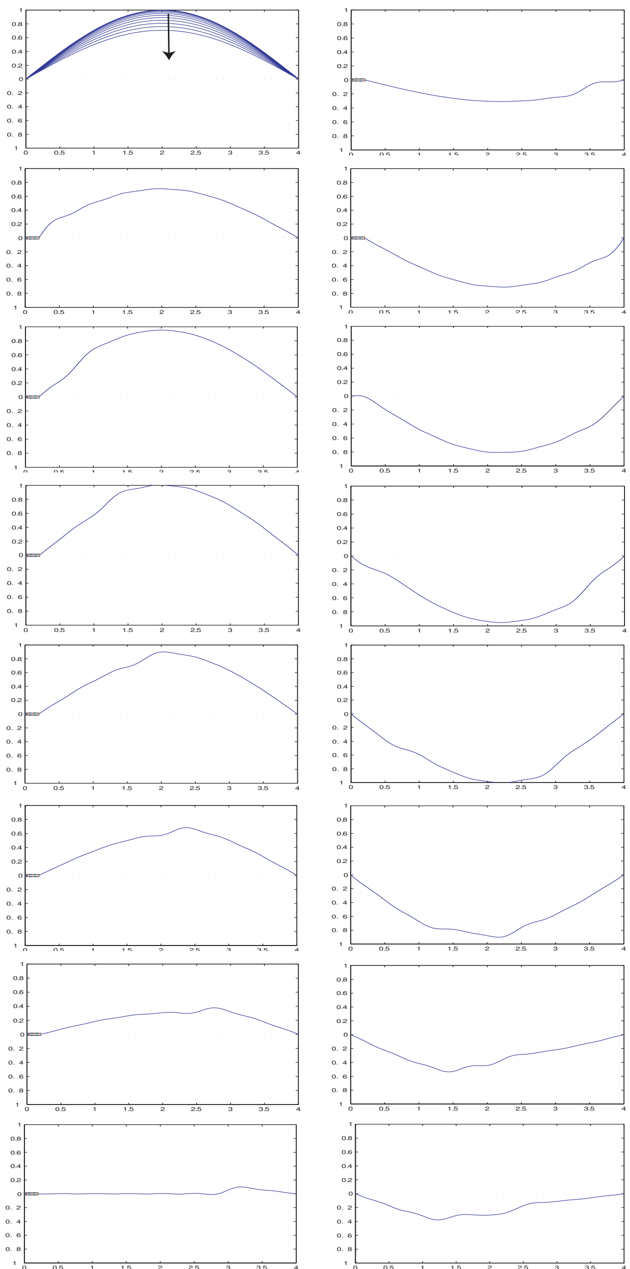

Figure 5. APPLICATION AND REMOVAL OF CONSTRAINT WITH $x_o = 0.05l$ AT $t_o = 1.5$ SEC, THE PERCENTAGE OF THE WORK DONE IS 12.8360 % .

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