Analytical heat conduction solution for two-dimensional Cartesian slab under the effect of laser pulse

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Abstract

The present work shows an easy and elegant analytical solution of two-dimensional transient heat conduction in two slabs bounded to each other and subjected to a pulse of laser beam. The heat transfer coefficients of the two slabs assumed to be change with direction. Separation of variables method used in the solution since of it easiness and its effectiveness with these kinds of problems. The solution compared with the data obtained by the numerical solution of the same problem by using COMSOL multiphysics 5.2. The data show a good agreement with the numerical solution, and show the behavior of the heat spreading within time in the domain

Symbol	Description	units
a_1, a_2	Heat conduction coefficient ratio.	--
b_1, b_2	Coefficients.	
C_n	Coefficient of integration.	
c_p	Heat transfer capacity.	$J/_{kg.K}$
d, l	Dimensions.	m
F	Pulse parameter.	
h	Heat convection coefficient.	$\frac{W}{m^2.K}$
k_h, k_v	Directional heat conduction coefficient.	$\frac{W}{m.K}$
T	Temperature.	K

Greek symbols

α	Heat transfer diffusivity.	
$\beta, \gamma, \eta, \lambda$	Eigen parameters.	
ϵ	Thickness of laser beam	m

1- Introduction:

A great attention given to study multi-layers combinations of different materials due to their wide applications in industry. Heat conduction takes greatest share of that attention for these combinations weather the materials were isotropic or anisotropic, steady or transient. Therefore, many kinds of analytical and numerical solutions appeared due to that attention. Focus on analytical solutions taken, since present work is in that field. One can find books dealing with kinds of analytical solutions for enormous cases [1, and 2]. However, accurate details in the application cases led to

different kinds of solutions in literatures for each case. Weather solutions were steady [3, 4, and 5], or transient [6, 7, 8, 9, 10, 11, 12, and 13], Cartesian [3, 5, 7, 8, 10, and 13], cylindrical [4, 6, 11, and 12], or spherical coordinates [9], analytical solution changes according to nature of each case. Solution may be based on series [5, 9, and 13], transformation [3, 6, 7, 8, and 11], or by using separation of variables [3, 10, and 12]. However, it is appeared hear that analytical solutions need mathematical skills to conduct the complex solution of complex geometries or boundary and initial conditions. Nevertheless, still, these kind of solutions are attractive due to their elegance and accuracy.

In present work, anisotropic, two-dimensional, Cartesian, two-layered slab taken under influence of a pulse of laser beam in center of that slab as shown in figure 1. Solution conducted is analytical by the use of separation of variables, where the solution compared to numerical procedure to validate the solution.

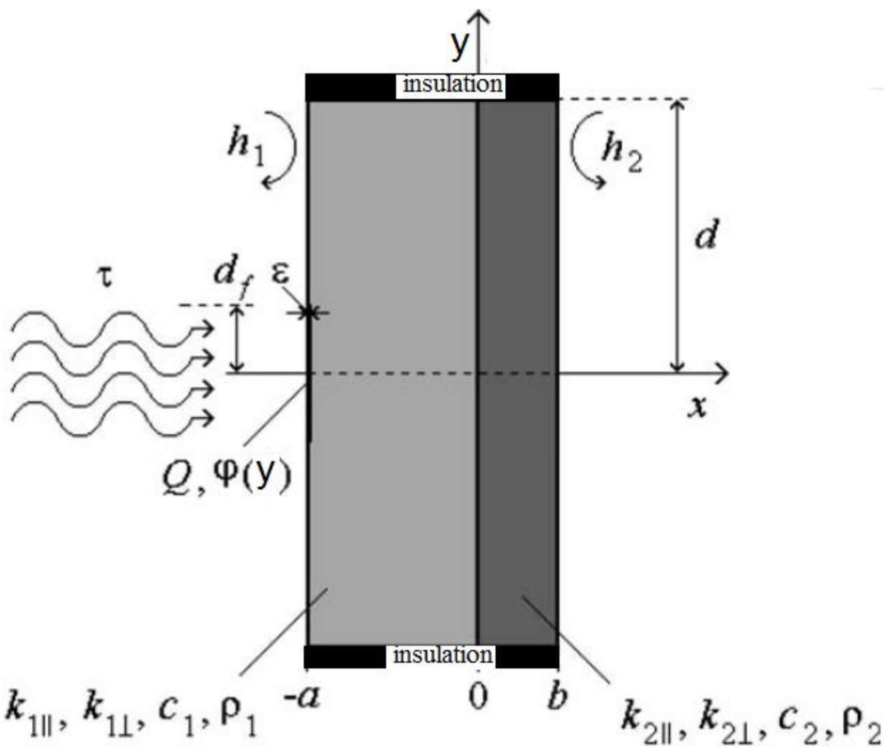


Figure 1 Problem of the present work.

2- Mathematical analysis:

The problem solved analytically starting from basic equations of Fourier's equation of conduction heat transfer for two- dimensional Cartesian coordinates:

$$\frac{\partial}{\partial x}\left(k_{h1}\frac{\partial T_1}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_v\frac{\partial T_1}{\partial y}\right)=\rho_1 c_{p1}\frac{\partial T_1}{\partial t} \tag{1}$$

$$\frac{\partial}{\partial x}\left(k_{h2}\frac{\partial T_2}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_v\frac{\partial T_2}{\partial y}\right)=\rho_1 c_{p1}\frac{\partial T_2}{\partial t} \tag{2}$$

Note that heat transfer coefficient anisotropic with direction.

Boundary and initial conditions according to the problem are:

$$k_{h1}\frac{\partial T_1(-l.y.t)}{\partial x}=h_1 T_1(-l.y.t) \tag{3}$$

$$k_{h2}\frac{\partial T_2(b.y.t)}{\partial x}=h_2 T_2(b.y.t) \tag{4}$$

$$k_{h1}\frac{\partial T_1(0.y.t)}{\partial x}=k_{h2}\frac{\partial T_2(0.y.t)}{\partial x} \tag{5}$$

$$T_1(0.y.t)-T_2(0.y.t)=2bk_{h1}\frac{\partial T_1(0.y.t)}{\partial x} \tag{6}$$

$$\frac{\partial T_1(x.0.t)}{\partial y}=0 \tag{7}$$

$$\frac{\partial T_2(x.0.t)}{\partial y}=0 \tag{8}$$

$$\frac{\partial T_1(x.dt)}{\partial y}=0 \tag{9}$$

$$\frac{\partial T_2(x.d.t)}{\partial y}=0 \tag{10}$$

With initial conditions:

$$T_2(x.d.0)=0 \tag{11}$$

$$T_1(x.0.t)=\begin{cases}\emptyset(y)=F & 0\le y\le \varepsilon\\ 0 & \varepsilon\le y\le d\end{cases} \tag{12}$$

The above initial condition (12) assumes opaque slab and no penetration in deep layers of slab. In addition, initial condition is not continuous where a sharp change in formula at boundaries. For this reason, the formula changed by an equivalent Fourier series as shown below:

$$\phi(y)=F\left[\frac{2\varepsilon}{d}+\frac{2}{\pi}\sum_{m=1}^{m=\infty}\frac{\sin(\frac{2m\varepsilon}{d})}{m}\cos(\frac{m\pi y}{d})\right] \qquad (12)$$

In order to solve differential equations (1) and (2), separation of variables proposed by following assumption:

$$T_1(x.y.t)=X_1(x).Y_1(y).\Gamma_1(t) \tag{13}$$

$$T_2(x.y.t) = X_2(x).Y_2(y).\Gamma_2(t) \tag{14}$$

Substation of assumptions (13) and (14) in differential equations led to:

$$a_1 \frac{X_1''}{X_1} + \frac{Y_1''}{Y_1} = \frac{1}{\alpha_{v1}} \frac{\Gamma_1'}{\Gamma_1} \tag{15}$$

$$a_2 \frac{X_2''}{X_2} + \frac{Y_2''}{Y_2} = \frac{1}{\alpha_{v2}} \frac{\Gamma_2'}{\Gamma_2} \tag{16}$$

Where:

$$a_1 = \frac{k_{h1}}{k_{v1}} \tag{17}$$

$$a_2 = \frac{k_{h2}}{k_{v2}} \tag{18}$$

$$\alpha_{v1} = \frac{k_{v1}}{\rho_1 c_{p1}} \tag{19}$$

$$\alpha_{v2} = \frac{k_{v2}}{\rho_2 c_{p2}} \tag{20}$$

In order to solve differential equations (15) and (16), Eigen parameters introduced to help in solution, which leads to the following sub-differential equations:

$$\alpha_{v1} \frac{\Gamma_1'}{\Gamma_1} = -\beta^2 \tag{21}$$

$$\alpha_{v2} \frac{\Gamma_2'}{\Gamma_2} = -\beta^2 \tag{22}$$

$$\frac{X_1''}{X_1} = -\gamma^2 \tag{23}$$

$$\frac{X_2''}{X_2} = -\gamma^2 \tag{24}$$

$$\frac{Y_1''}{Y_1} = -\lambda^2 \tag{25}$$

$$\frac{Y_2''}{Y_2} = -\eta^2 \tag{26}$$

Where:

$$-\lambda^2 = \frac{-\beta^2}{\alpha_{v1}} + a_1 \gamma^2 \tag{27}$$

$$-\eta^2 = \frac{-\beta^2}{\alpha_{v2}} + a_2\gamma^2 \tag{28}$$

Solutions of differential equations (21-26) are:

$$\Gamma_1 = C_1 e^{-\frac{\beta^2}{\alpha_{v1}}t} \tag{29}$$

$$\Gamma_2 = C_2 e^{-\frac{\beta^2}{\alpha_{v2}}t} \tag{30}$$

$$X_1 = C_3 \sin(\gamma x) + C_4\cos(\gamma x) \tag{31}$$

$$X_2 = C_5 \sin(\gamma x) + C_6\cos(\gamma x) \tag{32}$$

$$Y_1 = C_7 \sin(\lambda y) + C_8\cos(\lambda y) \tag{33}$$

$$Y_2 = C_9 \sin(\eta y) + C_{10}\cos(\eta y) \tag{34}$$

Substitution of boundary conditions (7-10) in (33) and (34) lead to:

$$Y_1 = \sum_{n=0}^{n=\infty} C_{8n}\sin(\lambda_n y) \tag{35}$$

$$Y_2 = \sum_{n=0}^{n=\infty} C_{9n}\sin(\eta_n y) \tag{36}$$

Where Eigen values:

$$\lambda_n = \frac{(2n+1)\pi}{2} \tag{37}$$

$$\eta_n = \frac{(2n+1)\pi}{2} \tag{38}$$

Substitution of boundary conditions (3-6) in differential equations (31) and (32) lead to:

$$X_1 = C_4[b_1\, sin(\gamma x) + cos(\gamma x)] \tag{39}$$

$$X_2 = C_5[sin(\gamma x) + b_2 cos(\gamma x)] \tag{40}$$

Where:

$$b_1 = \frac{h_1\, cos(\gamma l) + k_{h1}\gamma sin(\gamma l)}{h_1\, sin(\gamma l) + k_{h1}\gamma cos(\gamma l)} \tag{41}$$

$$b_2 = \frac{\gamma\, cos(\gamma b) + \frac{h_2}{k_{h1}} sin(\gamma b)}{\gamma\, sin(\gamma b) - \frac{h_2}{k_{h1}} cos(\gamma b)} \tag{42}$$

Application of boundary conditions (3) and (4), lead to:

$$C_4 = \frac{2bb_1a_1\gamma k_{h1}}{b_2 - a_1} \tag{43}$$

$$C_5 = \frac{2bb_1\gamma k_{h1}}{b_2 - a_1} \tag{44}$$

Simultaneous solution of Eigen parameters (27) and (28) gave:

$$\beta^2 = \left[\frac{a_2}{a_2}\lambda^2 - \eta^2\right]\left[\frac{a_1\alpha_{v1}\alpha_{v2}}{a_2\alpha_{v2} - a_2\alpha_{v1}}\right] \tag{29}$$

$$\gamma^2 = \frac{\beta^2}{a_1\alpha_{v1}} - \frac{\lambda^2}{a_1} \tag{30}$$

The final solution became:

$$T_1(x.y.t) = \sum_{n=1}^{n=\infty} C_{11n} sin(\lambda_n y)(b_1\, sin(\gamma_n x) + cos(\gamma_n x))e^{-\frac{\beta^2}{\alpha_{v1}}t} \tag{31}$$

$$T_2(x.y.t) = \sum_{n=1}^{n=\infty} C_{12n} sin(\lambda_n y)(sin(\gamma_n x) + b_2 cos(\gamma_n x))e^{-\frac{\beta^2}{\alpha_{v2}}t} \tag{32}$$

Where:

$$C_{11n} = C_1 C_{8n} C_{4n} \tag{33}$$

$$C_{12n} = C_2 C_{5n} C_{9n} \tag{34}$$

Substitution of initial condition (12) in (31) as well as using orthogonally gave:

$$C_{11n} = \frac{\int_{-l}^{0}\int_{0}^{d} \phi(y)\, sin(\lambda_n)(b_1\, sin(\gamma_n x) + cos(\gamma_n x))dy\, dx}{\int_{-l}^{0}\int_{0}^{d} sin^2(\lambda_n y)(b_1\, sin(\gamma_n x) + cos(\gamma_n x))^2 dydx} \tag{35}$$

solution of above double integration is:

$$C_{11n} = \frac{b_1\left(\cos(\gamma_n l) + \frac{1}{b_1}\sin(\gamma_n l) - 1\right)}{\gamma_n d\lambda_n(C_{13n} + C_{14n}\sin(\gamma_n l) + \cos(\gamma_n l))}[2\lambda_n \cos(\lambda_n d)]\left[\frac{F\epsilon}{d\lambda_n^2}(sec(\lambda_n d) - 1) + \frac{P_1(1 + sec(\lambda_n d))}{\frac{\pi}{d} - \lambda_n^2} + \frac{P_2(sec(\lambda_n d) - 1)}{\frac{4\pi^2}{d^2} - \lambda_n^2} + \frac{P_3(1 + sec(\lambda_n d))}{\frac{9\pi^3}{d^3} - \lambda_n^2}\right] \tag{36}$$

Where:

$$C_{13n} = b_1 - \frac{l\gamma_n}{2}(b_1^2 + 1) \tag{37}$$

$$C_{14} = 1 - \frac{b_1^2}{2} \tag{38}$$

$$P_1 = \frac{2F}{\pi}\sin(\frac{\pi\epsilon}{d}) \tag{39}$$

$$P_2 = \frac{F}{\pi}\sin(\frac{2\pi\epsilon}{d}) \tag{40}$$

$$P_3 = \frac{2F}{3\pi}\sin(\frac{3\pi\epsilon}{d}) \tag{41}$$

same principle of orthogonally lead to:

$$C_{12n} = \frac{C_{11n}(1-b_1\gamma_n)}{b_2} \quad (42)$$

Numerical solution:

In order to validate analytical solution, problem solved numerically using COMSOL Multiphysics 5.2 program. A transient solution of heat transfer in solids used. An extra fine mesh for element size chosen with Physics-controlled mesh as shown below.

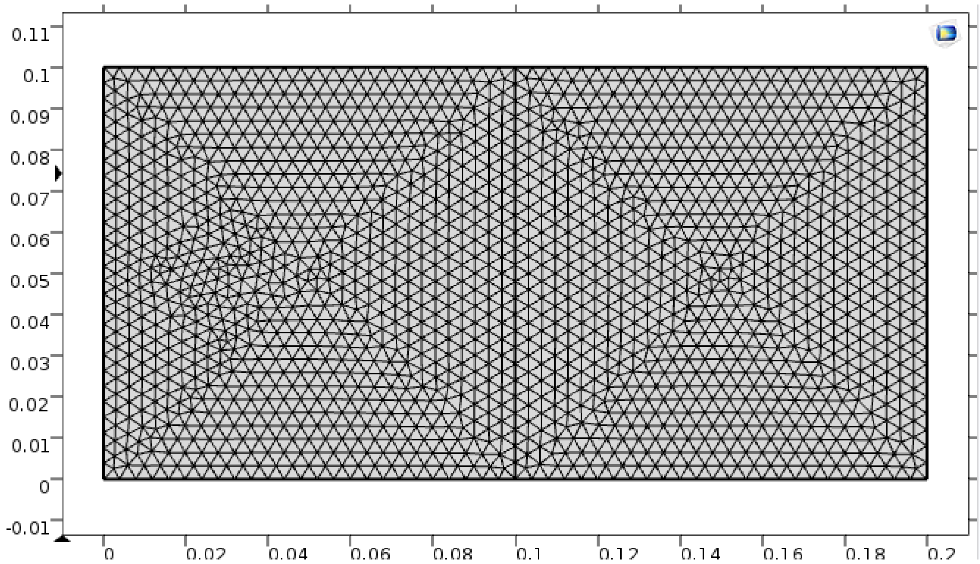


Figure 2 Mesh of numerical solution.

Time interval for transient solution is 0.01 sec. Conduction heat transfer coefficient taken constant with direction, for simplicity of the solution. The left slab taken to be Aluminum while the other is Copper. Heat transfer parameters assumed constant. Pulse of heat input to medium taken to be 30kW of heat as initial condition of the problem. Table below shows comparison of numerical solution with that found numerically. Data taken for a point in the middle of domain of x=0, and y=0.5, for periods of time after applying initial condition.

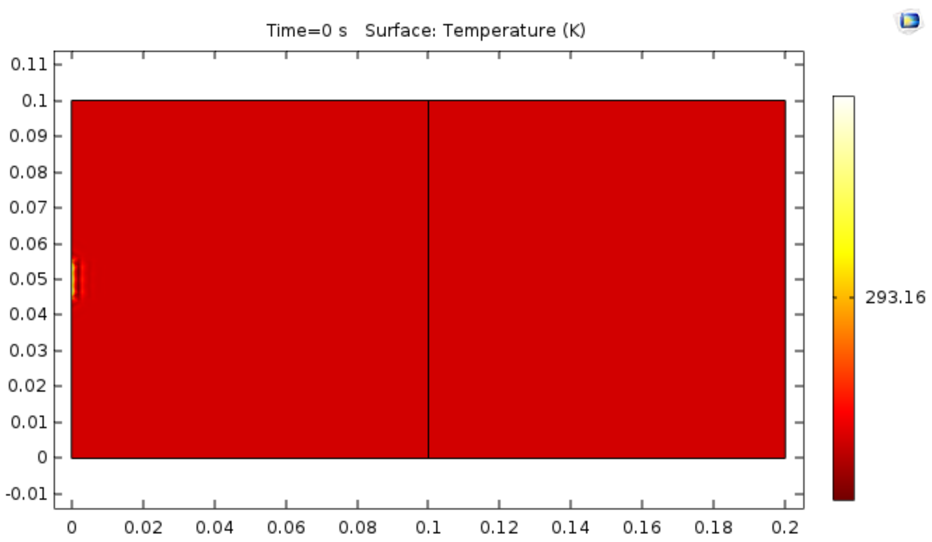
time	Numerical solution	Analytical solution
0	293.15	293.15
30	293.16	293.158
60	293.17	293.169
90	293.17	293.169

120	293.17	293.169

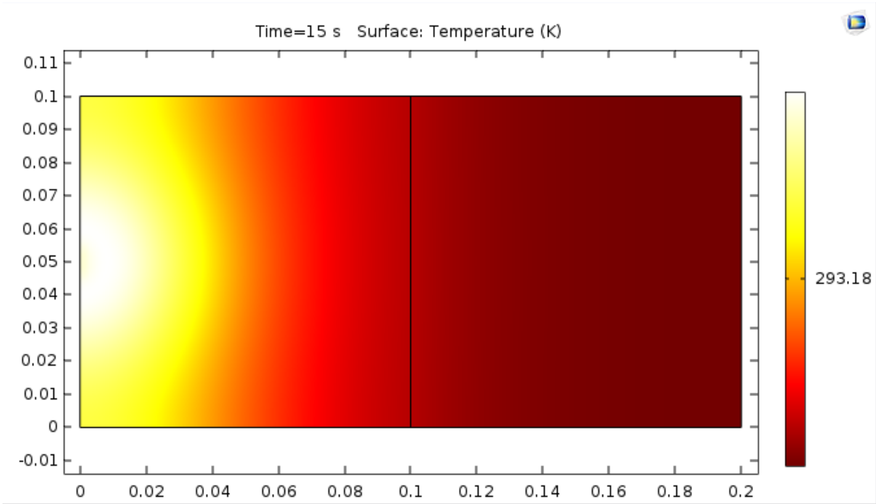
Data show a slight increase in temperature, and a settling after 30 seconds. The most important thing is closeness of numerical and analytical solutions.

Results and discussions:

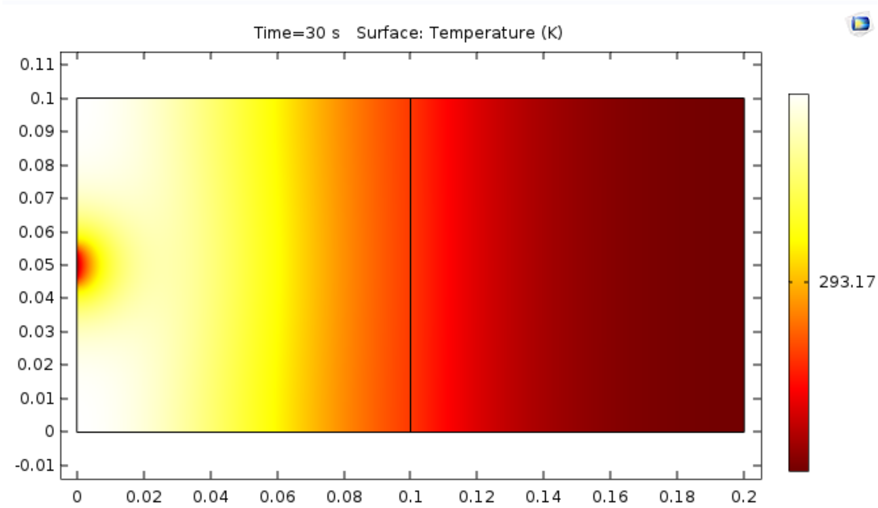
Good agreement of analytical solution with that of numerical suggest showing more transient results of domains of solution. Figures 3 (a-f) show transient temperature distribution with a time interval of 15 seconds. Heat spread in as half circular shape within first interval, then isothermal lines starts to progress in a shape of parallel line due to insulation of top and bottom boundaries. After a minute and a half, first domain show to reach a uniform temperature distribution, while the second domain still suffering from heat progress in parallel isotherms. Whole process does not going out of common sense about the behavior of temperature progress, where results are only shown to expand the base of data presented. Solution may extended, in future work, for a multiple laser pulses, or to use materials have a temperature dependent heat transfer coefficients.



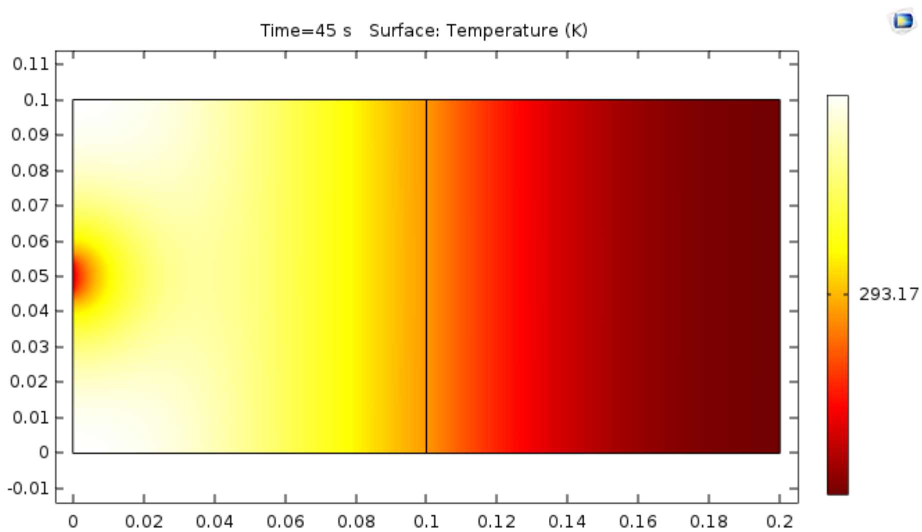
(a)


Time=15 s Surface: Temperature (K)
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-0.01
0 0.02 0.04 0.06 0.08 0.1 0.12 0.14 0.16 0.18 0.2
293.18

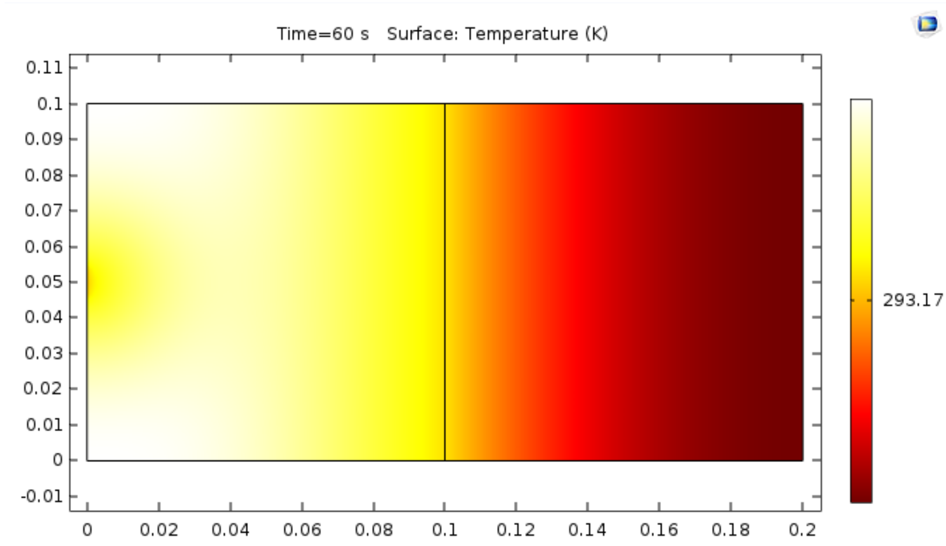
(b)


Time=30 s Surface: Temperature (K)
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-0.01
0 0.02 0.04 0.06 0.08 0.1 0.12 0.14 0.16 0.18 0.2
293.17

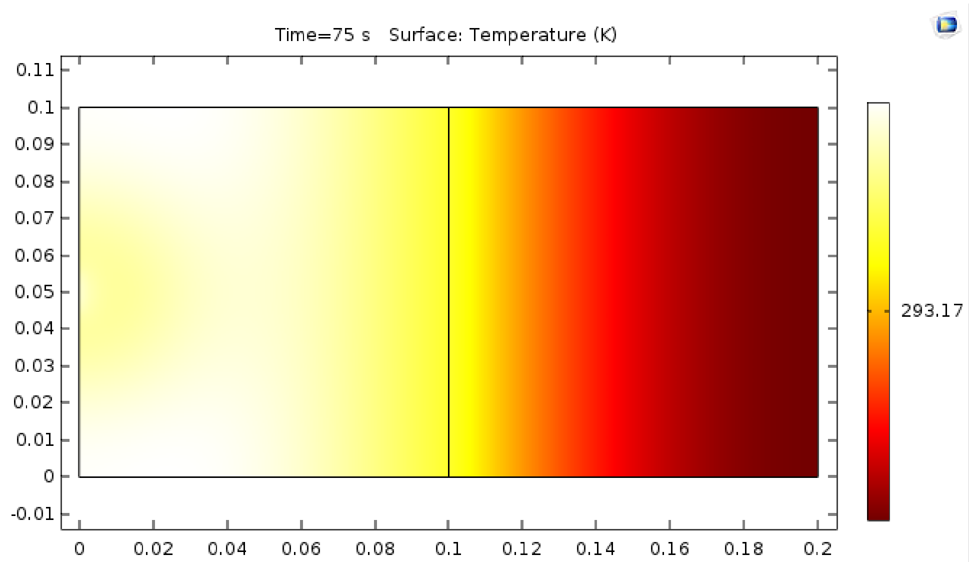
(c)


Time=45 s Surface: Temperature (K)
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-0.01
0 0.02 0.04 0.06 0.08 0.1 0.12 0.14 0.16 0.18 0.2
293.17

(d)



(e)



(f)

Figure 3(a-f) Transient temperature distribution for two domains of problem.

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