
Chapter One/Introduction to Compressible Flow

1.1. Introduction

In general flow can be subdivided into:

i. Ideal and real flow.

For ideal (inviscid) flow viscous effect is ignored. The momentum equations are Euler's equations that derived in 1755 by Euler.

For real (viscose) viscous effect is considered. The momentum equations are Navier-Stokes equations.

ii. Steady and unsteady flow.

For steady flow, flow properties are time independent and mass exits from the system equals the mass enters the system.

For unsteady, flow properties are time dependent and mass exit s from the system may or may not equals the mass enters the system and the difference causes system mass change.

iii. Compressible and incompressible flow

For compressible flow, density becomes an additional variable; furthermore, significant variations in fluid temperature may occur as a result of density or pressure changes. There are four possible unknowns, and four equations are required for the solution of a problem in compressible gas dynamics: equations for the conservation of mass, momentum, and energy, and a thermodynamic relations and equation of state for the substance involved. The study of compressible flow necessarily involves an interaction between thermodynamics and fluid mechanics.

For incompressible flow can be assumed with density is not a variable. For this type of flow, two equations are generally sufficient to solve the problems encountered: the continuity equation or conservation of mass and a form of the Bernoulli equation, derivable from either momentum or energy considerations. Variables are generally pressure and velocity.

iv. One, two and three-Dimensional Flow

One-dimensional flow, by definition, did not consider velocity components in the y or z directions, as in Figure (1.1a). In true one-dimensional flow, area changes are not allowed. For inviscid flow the velocity profile is shown in section (a) and (c). However, the more gradual the area change with x, the more exact becomes the one-dimensional approximation.

For viscose flow the velocity profiles is shown in Figure (1.1b). Actually, due to viscosity, the flow velocity at the fixed wall must be zero as in sections (a) and (c).

Consider the flow in a varying area channel. The velocity profile in a real fluid is shown in Figure (1.1b) section (b).

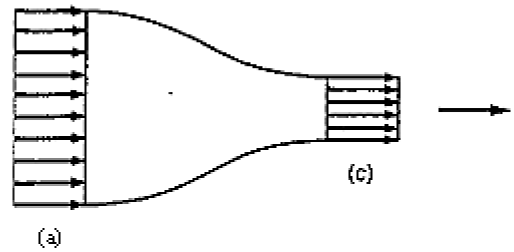


Figure 1.1a: One dimension flow

A complete solution of a problem in a fluid mechanics requires a three-dimensional analysis. However, even for incompressible flow a complete solution in three dimensions is possible only numerically with the aid of computer programs. Fortunately, a great many compressible flow problems can be solved with the use of a one-dimensional analysis. One-dimensional flow implies that the flow variables are functions of only one space coordinate.

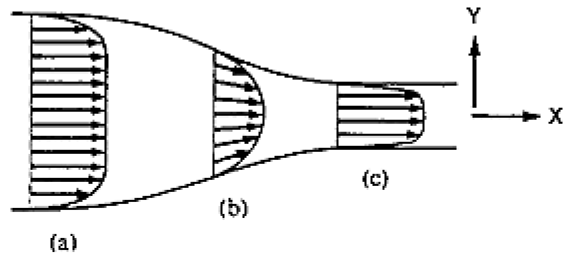


Figure 1.1b: Real flow in varying area duct

1.2. Control volume approach

Figure (1.2) shows an arbitrary mass at time t and the same mass at time $t + \Delta t$, which composes the same mass particles at all times. If Δt is small, there will be an overlap of the two regions as shown, with the common region identified as region 2. At time t the given mass particles occupy regions 1 and 2. At time $t + \Delta t$ the same mass particles occupy regions 2 and 3. Regions 1 & 2, which originally confines of the mass, are called the ***control volume***.

Introducing of concept of ***material derivative*** of any ***extensive property*** (a property which is mass dependent such as mass, enthalpy, internal energy … etc) transforms to a control volume approach gives a valuable general relation called ***Reynolds's Transport Theorem*** that can be used to find property change for many particular situations. Let

$X\ (pronounce\ chi) \equiv$ the total amount of any extensive property in a given mass.

$x \equiv$ the amount of X per unit mass. Thus

$$X = \int x\, dm = \iiint\limits_{c.v.} x\rho d\Upsilon$$

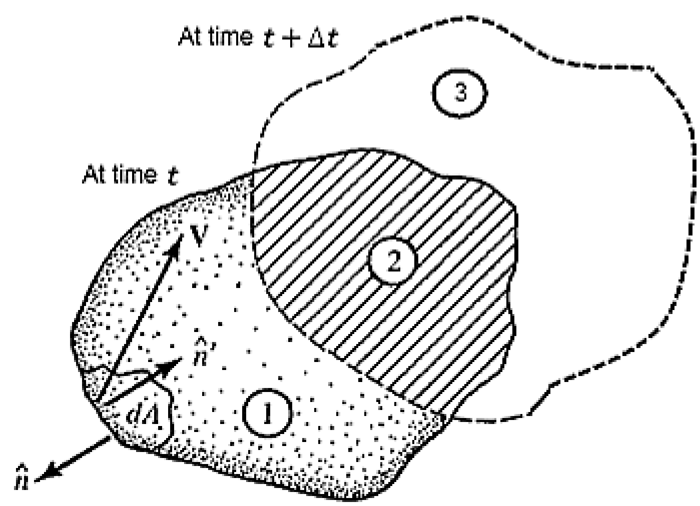


Figure 1.2: Flow into control volume.

We construct our material derivative from the mathematical definition

$$\frac{DX}{dt} = \lim_{\Delta t \to 0}\left[\frac{(final\ value\ of\ X)_{t+\Delta t} - (initial\ value\ of\ X)_t}{\Delta t}\right]$$

$$\frac{DX}{dt} = \lim_{\Delta t \to 0}\left[\frac{(X_2 + X_3)_{t+\Delta t} - (X_1 + X_2)_t}{\Delta t}\right] \qquad (1.1)$$

Now for the term

$$\lim_{\Delta t \to 0}\frac{(X_3)_{t+\Delta t}}{\Delta t}$$

The numerator represents the amount of X in region 3 at time $(t + \Delta t)$, and by definition *region 3 is formed by the fluid moving out of the control volume*. Then;

$$\lim_{\Delta t \to 0}\frac{(X_3)_{t+\Delta t}}{\Delta t} = \iint_{cs,out} x\,\rho\,(\boldsymbol{V}.\hat{n})\,dA \approx total\ amount\ of\ X\ in\ region\ 3 \qquad (1.2)$$

This integral is called a *flux* or *rate* of X flow *out* of the control volume.

Now let us consider the term

$$\lim_{\Delta t \to 0}\frac{(X_1)_t}{\Delta t}$$

Region 1 has been formed by the original mass particles moving into the control volume (during time Δt). Thus

$$\lim_{\Delta t \to 0}\frac{(X_1)_t}{\Delta t} = \iint_{cs,in} x\,\rho\,(\boldsymbol{V}.\breve{n})\,dA \approx total\ amount\ of\ X\ in\ region\ 1 \qquad (1.3)$$

This integral is called a *flux* or *rate* of X flow *into* the control volume.

Now look at the first and last terms of equation (1.1) which is:

$$\lim_{\Delta t \to 0}\left[\frac{(X_2)_{t+\Delta t} - (X_2)_t}{\Delta t}\right] = \frac{\partial X_{c.v.}}{\partial t} = \frac{\partial}{\partial t}\iiint_{cv} x\,\rho\,d\Upsilon \qquad (1.4)$$

Note that the partial derivative notation is used since the region of integration is fixed and time is the only independent parameter allowed to vary. Also note that as Δt approaches zero, region 2 approaches the original control volume. Then eq. (1.1) becomes

$$\frac{DX}{dt} = \lim_{\Delta t \to 0}\left[\frac{(X_2 + X_3)_{t+\Delta t} - (X_1 + X_2)_t}{\Delta t}\right]$$

$$= \frac{\partial}{\partial t}\iiint_{cv} x\,\rho\,d\Upsilon + \iint_{cs,out} x\,\rho\,(\boldsymbol{V}.\hat{n})\,dA - \iint_{cs,in} x\,\rho\,(\boldsymbol{V}.\breve{n})\,dA \qquad (1.5)$$

As $\hat{n} = -\breve{n}$ then the last two terms become

$$\iint_{cs,out} x\,\rho\,(\boldsymbol{V}.\hat{n})\,dA - \iint_{cs,in} x\,\rho\,(\boldsymbol{V}.\breve{n})\,dA = \iint_{cs} x\,\rho\,(\boldsymbol{V}.\hat{n})\,dA$$

which is the net rate of X passes the control volume surface. The final transformation becomes:

$$\left(\frac{DX}{Dt}\right) = \frac{\partial}{\partial t}\iiint_{cv} x\,\rho\,d\Upsilon + \iint_{cs} x\,\rho\,(\boldsymbol{V}.\hat{n})\,dA \qquad (1.6)$$

This relation, known as ***Reynolds's Transport Theorem***, which can be interpreted in words as: The rate of change of X property for a fixed mass system of fluid particles as it is moving is equal to the rate of change of X inside the control volume *plus* the *net* efflux of X from the control volume (flow out minus flow in across control volume boundary).

Where

$\frac{D}{Dt}$: Material or total or substantial derivative

$\frac{\partial}{\partial t}$: Partial derivative with respect to time

cv : control volume that containing the mass.

cs : control surface that surrounding the control volume.

X : Mass-dependent (extensive) property; scalar or vector quantity.

x : is the amount of the property per unit mass. For mass it equals one.

ρ : Fluid density (kg/m^3).

$d\Upsilon$: Infinitesimal (very small) control volume.

dA : Infinitesimal control surface.

$\boldsymbol{V}$: Velocity vector.

$\hat{n}$: Outward unit vector which is perpendicular to dA.

$\breve{n}$: Inward unit vector which is perpendicular to dA.

Examples of the application of this powerful transformation equation are conservation of mass, energy and momentum equations which are presented in the next chapter.

References:

1. James John & Thie Keith, Gas dynamics, 3td edition, Pearson prentice hall, Upper Saddle, New Jersey, 2006.
2. Robert D. Zucker & Oscar Biblarz , Fundamental of Gas Dynamics, John Wily & Sons, New York, 2002.
3. منذر اسماعيل الدروبي، مبادئ ديناميك الغازات، بغداد، وزارة التعليم العالي و البحث العلمي، 1980.

Chapter Two/Basic Equation of Compressible Flow

2.1. Conservation of mass:

$$\left(\frac{D\mathrm{X}}{Dt}\right) = \frac{\partial}{\partial t}\iiint_{cv} \chi\, \rho\, d\Upsilon + \iint_{cs} \chi\, \rho\, (\boldsymbol{V}.\hat{n})\, dA$$

Let $\mathrm{X} \equiv mass$ so $\chi = 1$. For fixed amount of mass that moves through the control volume:

$$\left(\frac{DMass}{Dt}\right) = 0 \qquad (2.1)$$

And for steady flow:

$$\frac{\partial}{\partial t}\iiint_{cv} \rho\, d\Upsilon = 0 \qquad (2.2)$$

So the second term must equals to zero.

$$\iint_{cs} \rho\, (\boldsymbol{V}.\hat{n})\, dA = 0 \qquad (2.3)$$

Let us now evaluate the remaining integral for the case of one-dimensional flow. Figure (2.1) shows fluid crossing a portion of the control surface. Recall that for one-dimensional flow any fluid property will be constant over an entire cross section. Thus both the density and the velocity can be brought out from under the integral sign. If the surface is always chosen perpendicular to V, the integral is very simple to evaluate:

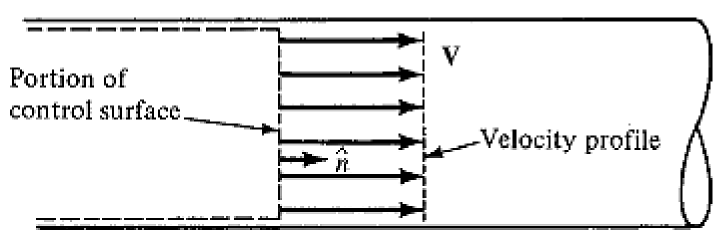


Figure 2.1: One-dimensional velocity profile.

$$\int \rho\, (\boldsymbol{V}.\hat{n})\, dA = \rho V \int dA = \rho\, V\, (A_e - A_i) \qquad (2.4)$$

But integral in eq. 2.3 must be evaluated over the entire control surface, which yields:

$$\iint_{cs} \rho\ (\boldsymbol{V}.\hat{n})\ dA = \sum \rho\ V\ A \qquad (2.5)$$

This summation is taken over all sections where fluid crosses the control surface. It is positive where fluid leaves the control volume (since $\boldsymbol{V}.\hat{n}$ is positive here) and negative where fluid enters the control volume.

For steady, one-dimensional flow, the continuity equation for a control volume becomes:

$$\sum \rho\ V\ A = 0 \qquad (2.6)$$

If there is only one section where fluid enters and one section where fluid leaves the control volume, this becomes:

$$(\rho\ V\ A)_{out} = (\rho\ V\ A)_{in} \qquad (2.7)$$

$$\dot{m} = \rho\ V\ A = const \qquad (2.8)$$

V is the component of velocity perpendicular to the area A. If the density ρ is in kg/m^3, the area A is in m^2 and velocity V is in m/s, then $\dot{m}$ is in kg/s.

Note that *as a result of steady flow* the mass flow rate into a control volume is equal to the mass flow rate out of the control volume. But if the mass flow rates into and out of a control volume is the same it doesn’t ensure that the flow is steady.

For steady one-dimensional flow, differentiating eq. 2.8 gives:

$$d(\rho\ V\ A) = 0 = V\ A\ d(\rho) + \rho\ V d(\ A) + \rho\ A\ d(V) \qquad (2.9)$$

Dividing by $\rho\ V\ A$

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \qquad (2.10)$$

This expression can also be obtained by first taking the natural logarithm of equation (2.8) and then differentiating the result. This is called *logarithmic differentiation*.

This differential form of the continuity equation is useful in interpreting the changes that must occur as fluid flows through a duct, channel, or stream-tube. It indicates that if mass is to be conserved, the changes in density, velocity, and cross sectional area must compensate for one another. For example, if the area is constant *($dA = 0$)*, any increase in velocity must be accompanied by a corresponding decrease in density. We shall also use this form of the continuity equation in several future derivations.

2.2. Conservation of energy.

From first law of thermodynamics

$$Q = W + \Delta E \qquad (2.11)$$

Where ΔE is the change in total energy of the system i.e. it is the change in internal, kinetic and potential energies, $\Delta(U + K.E. + P.E.)$. Eq. 2.11 can be written on a rate basis to yield an expression that is valid at any instant of time:

$$\frac{\delta Q}{dt} = \frac{\delta W}{dt} + \frac{dE}{dt} \qquad (2.12)$$

$\delta Q/dt$ and $\delta W/dt$ represent instantaneous rates of heat and work transfer between the system and the surrounding. They are rates of energy transfer across the boundaries of the system. These terms are *not* material derivatives since heat and work are not properties of a system. On the other hand, energy is a property of the system and dE/dt is a material derivative, then:

$$\left(\frac{DE}{Dt}\right) = \frac{\partial}{\partial t}\iiint_{cv} e\,\rho\,d\Upsilon + \iint_{cs} e\,\rho\,(\boldsymbol{V}.\hat{n})\,dA \qquad (2.13)$$

For one-dimensional, steady flow the last integral is simple to evaluate, as $e, \rho, and\ V$ are constant over any given cross section. Assuming that the velocity V is perpendicular to the surface A, we have

$$\iint_{cs} e\,\rho\,(\boldsymbol{V}.\hat{n})\,dA = \sum(\rho\,V\,A)\,e \sum \dot{m}\,e \qquad (2.14)$$

$$\frac{\partial}{\partial t}\iiint_{cv} e\,\rho\,d\Upsilon = 0 \qquad (2.15)$$

We must be careful to include all forms of work, whether done by pressure forces or shear forces. Figure (2.2) shows a simple control volume. Note that the control surface is chosen carefully so that there is no fluid motion at the boundary, except:

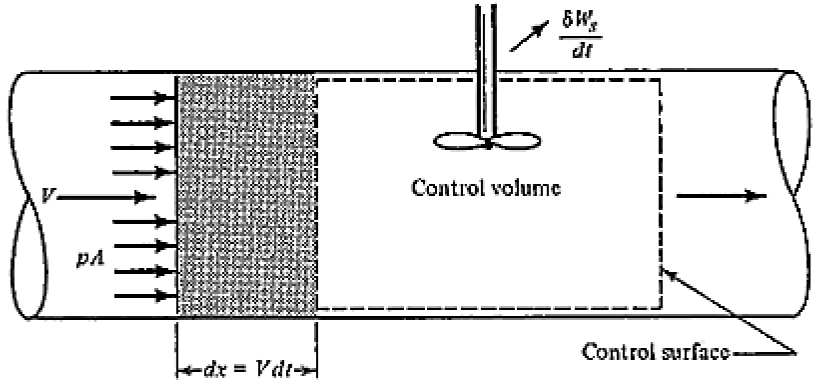


Figure 2.2: Identification of work quantities.

(a) Fluid enters and leaves the system.

(b) A mechanical device crosses the boundaries of the system.

For fluid enters and leaves the system, the pressure forces do work to push fluid into or out of the control volume. The shaded area at the inlet represents the fluid that enters the control volume during time dt. The work done here is:

$$\delta \acute{W} = F.dx = p\,A\,dx = p\,A\,V\,dt \qquad (2.16)$$

The rate of doing work, which called *flow work*, is

$$\frac{\delta \acute{W}}{dt} = pAV = \dot{m}pv \qquad (2.17)$$

The rate at which work is transmitted out of the system by the mechanical device is $\delta W_s/dt$ and

$$\frac{\delta W}{dt} = \frac{\delta W_s}{dt} + \frac{\delta \acute{W}}{dt} = \frac{\delta W_s}{dt} + \dot{m}pv \qquad (2.18)$$

Thus for steady one-dimensional flow the energy equation for a control volume becomes

$$\frac{\delta Q}{dt} = \frac{\delta W_s}{dt} + \sum \dot{m}(e + pv) \qquad (2.19)$$

The summation is taken over all sections where fluid crosses the control surface and is positive where fluid leaves the control volume and negative where fluid enters the control volume.

If there is only one section where fluid leaves and one section where fluid enters the control volume, we have, (from continuity), for steady flow:

$$\dot{m}_{in} = \dot{m}_{out} = \dot{m}$$

Let us take:

$$\frac{\delta Q}{dt} = \frac{\partial}{\partial t}\iiint_{cv} q\,\rho\,d\Upsilon + \iint_{cs} q\,\rho\,(\boldsymbol{V}.\hat{n})\,dA = \dot{m}q \qquad (2.20)$$

$$\frac{\delta W_s}{dt} = \frac{\partial}{\partial t}\iiint_{cv} w_s\,\rho\,d\Upsilon + \iint_{cs} w_s\,\rho\,(\boldsymbol{V}.\hat{n})\,dA = \dot{m}w_s \qquad (2.21)$$

Substitute in eqs (2.20) and (2.21) into eq (2.19) gives:

$$q = w_s + \sum(e + pv) \qquad (2.22)$$

$$q = w_s + \left(u + \frac{V^2}{2} + gz + pv\right)_{out} - \left(u + \frac{V^2}{2} + gz + pv\right)_{in} \qquad (2.23)$$

$$q = w_s + \left(h + \frac{V^2}{2} + gz\right)_2 - \left(h + \frac{V^2}{2} + gz\right)_1 \qquad (2.24)$$

This is the form of the energy equation that may be used to solve many problems. It is often referred as steady flow energy equation (SFEE).

For **unsteady flow**, since change of kinetic and potential energies within the system is negligible, then (Unsteady F.E. E) becomes:

$$\left\{Q + \left[\dot{m}\left(h + \frac{V^2}{2} + gz\right)\right]_{in}\right\} - \left\{W_s + \left[\dot{m}\left(h + \frac{V^2}{2} + gz\right)\right]_{out}\right\} = (\dot{m}u)_2 - (\dot{m}u)_1 \qquad (2.25)$$

$$\dot{m}_{out} - \dot{m}_{in} = \dot{m}_2 - \dot{m}_1 \qquad (2.26)$$

where u_2 and m_2 are internal energy and mass of the working fluid inside the system after change while u_1 and m_1 are internal energy and mass of the working fluid inside the system before change.

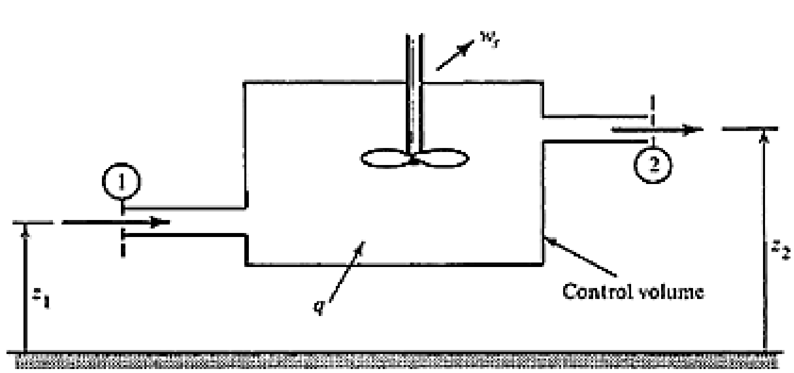


Figure 2.3: Finite control volume for energy analysis.

2.3. Conservation of momentum.

If we observe the motion of a given quantity of mass, Newton's second law tells us that the linear momentum will be changed in direct proportion to the applied forces. This is expressed by the following equation:

$$\sum \mathbf{F} = \frac{D(momentum)}{Dt} = \frac{\partial}{\partial t}\iiint_{cv} \mathbf{V}\,\rho\, d\Upsilon + \iint_{cs} \mathbf{V}\,\rho\,(\boldsymbol{V}.\hat{n})\,dA \qquad (2.27)$$

Here $\boldsymbol{V}$ besides it is a velocity vector it also represents the momentum per unit mass. This equation is usually called the *momentum* or *momentum flux equation*. $\sum \mathbf{F}$ represents the summation of all forces *on the fluid within the control volume* which maybe forces due to pressure, viscosity, gravity, surface tension … etc..

For steady flow the time rate of change of linear momentum stored inside the control volume is

$$\frac{\partial}{\partial t}\iiint_{cv} \boldsymbol{V}\,\rho\, d\Upsilon = 0 \qquad (2.28)$$

And momentum equation simplify to:

$$\sum \mathbf{F} = \iint_{cs} \mathbf{V}\,\rho\,(\boldsymbol{V}.\hat{n})\,dA \qquad (2.29)$$

The *x*-component of this equation would appear as

$$\sum F_x = \iint_{cs} V_x\,\rho\,V_x\,dA \qquad (2.30)$$

If there is only one section where fluid enters and one section where fluid leaves the control volume, we know (from continuity) that:

$\dot{m} = \dot{m}_{out} = \dot{m}_{in}$

And the momentum equation for a finite control volume becomes:

$$\sum F_x = \sum \dot{m}\,(V_{out} - V_{in}) \qquad (2.31)$$

The summation is taken over all sections where fluid crosses the control surface and is positive where fluid leaves the control volume and negative where fluid enters the control volume.

2.4. 1st law of thermodynamics.

First law of thermodynamics takes the following form

$$\sum Q = \sum W \tag{2.32}$$

Or

$$Q = W + \Delta E \tag{2.33}$$

First law of thermodynamics is a conservation of energy and we dealt with in 2.2.

2.5. 2nd law of thermodynamics.

Two concepts that are important to a study of compressible fluid flow are derivable from the second law of thermodynamics: the *reversible process* and the *property entropy*. For a thermodynamic system, *a reversible process is one after which the system can be restored to its initial state and leave no change in either system or surroundings*. As a consequence of this definition, it can be shown that a reversible process is quasi-static; changes occur infinitely slowly, with no energy being dissipated

Since thermodynamics, is a study of equilibrium states, definite thermodynamic equations for changes taking place during processes can be derived only for reversible processes; irreversible processes can only be described thermodynamically with the use of inequalities. Irreversible processes involve, for example, the following: friction, heat transfer through a finite temperature difference, sudden expansion, and magnetization with hysteresis, electrical resistance heating, and mixing of different gases.

In general, any natural process is irreversible, so the assumption of reversibility, while it may simplify the thermodynamic equations, necessarily yields an approximation. For many, cases, the assumption of reversibility leads to very accurate results; yet it is well to keep in mind that the reversible process is always an idealization.

The thermodynamic property derivable from the second law is entropy, which is-defined for a system undergoing a reversible process by $dS = (\delta Q/T)_{rev}$.

Entropy changes were defined in the usual manner in terms of reversible processes:

$$\Delta S = \int \frac{\delta Q_{Rev}}{T} \tag{2.34}$$

$$dS = dS_{external} + dS_{internal} \tag{2.35}$$

The term dS_e represents that portion of entropy change caused by the actual heat transfer between the system and its (external) surroundings. It can be evaluated readily from:

$$dS_e = \frac{\delta Q_{Rev}}{T} \tag{2.38}$$

One should note that dS_e can be either positive or negative, depending on the direction of heat transfer. If heat is removed from a system, δQ is negative and thus dSe will be negative. It is obvious that $dS_e = 0$ for an adiabatic process.

The term dS_i represents that portion of entropy change caused by irreversible effects. Moreover, dS_i effects are internal in nature, such as temperature and pressure gradients within the system as well as friction along the internal boundaries of the system. Note that this term depends on the process path and from observations we know that *all irreversibilities generate entropy* (i.e., cause the entropy of the system to increase). Thus we could say that

$$dS_i \geq 0 \tag{2.36}$$

Obviously, $dS_i = 0$ only for a reversible process. An isentropic process is one of constant entropy. This is also represented by $dS = 0$.

$$dS = 0 = dS_e + dS_i \tag{2.37}$$

A reversible-adiabatic process is isentropic, but an isentropic process does not have to be reversible and adiabatic we only know that $dS = 0$.

2.6. Equation of State.

An equation of state for a pure substance is a relation between pressure, .density, and temperature for that substance. Depending on the phase of the substance and on the range of conditions to which it is subjected, one of a number of different equations of state is applicable. However, for liquids or solids, these equations become so cumbersome and have such a limited range of application that it is generally more convenient to use tables of thermodynamic properties. For gases, an equation exists that does have a reasonably wide range of application, the *perfect gas law;* in its usual form, it is expressed as

$$p = \rho RT \tag{2.38}$$

For the derivation of the perfect gas law from kinetic theory, the volume of the gas molecules and the forces between the molecules are neglected. These assumptions are satisfied by a real gas only at very low pressures. However, even at reasonably high pressures, a real gas approximates a perfect gas as long as the gas temperature is great enough

2.7. Thermodynamics Relations.

Also the following relations are very useful equations. Starting with the thermodynamic property relation:

$$\delta q = du + \delta w \tag{2.39}$$

$$Tds = du + pdv = c_v dT + RT\frac{dv}{v} \tag{2.40}$$

$$Tds = dh - vdp = c_p dT - RT\frac{dp}{p} \tag{2.41}$$

For perfect gas with constant specific heats

$$\Delta s = c_v \int \frac{dT}{T} + R\int \frac{dv}{v} = c_v \ln T + R\ln v \tag{2.42}$$

$$\Delta s = c_p \int \frac{dT}{T} - R\int \frac{dp}{p} = c_p \ln T - R\ln p \tag{2.43}$$

$$R = c_p - c_v \qquad and \qquad \gamma = c_p/c_v$$

Example 2.1 Ten kilograms per second of air enters a tank $100\, m^3$ in volume while $2\, kg/s$ is discharged from the tank (Figure 2.4). If the temperature of the air inside the tank remains constant at $300\, K$, and the air can be treated as a perfect gas, find the rate of pressure rise inside the tank.

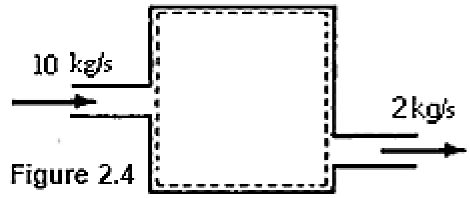


Figure 2.4

Solution:

Select a control volume as shown in the sketch. For this case the net rate of efflux of mass from the control volume is

$$\iint_{cs} \rho\, (\boldsymbol{V}.\hat{n})\, dA = -\,8\; kg/s$$

The volume is constant and also density is assumed constant inside the tank as temperature is constant, but it is time dependent.

$$0 = \frac{\partial \rho}{\partial t} \iiint_{cv} d\Upsilon + \iint_{cs} \rho\, (\boldsymbol{V}.\hat{n})\, dA$$

$$\iiint_{cv} d\Upsilon = \Upsilon = 100\, m^3$$

$$0 = 100 \frac{\partial \rho}{\partial t} - 8$$

From equation of state for a perfect gas

$$p = \rho R T$$

$$\frac{dp}{dt} = RT \frac{d\rho}{dt}$$

$$\frac{dp}{dt} = 287 * 300 * \frac{8}{100} = 6.888\;\; kPa/s$$

Example 2.2 Two kilograms per second of liquid hydrogen and eight kg/s of liquid oxygen are injected into a rocket combustion chamber in steady flow (Figure 2.5). The gaseous products of combustion are expelled at high velocity through the exhaust nozzle. Assuming uniform flow in the rocket nozzle exhaust plane, determine the exit velocity. The nozzle exit diameter is $30\; cm$. and the density of the gases at the exit plane is $0.18\; kg/m^3$

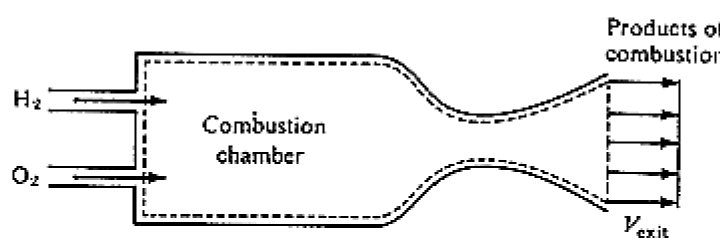


Figure 2.5

Solution

$$A = \frac{\pi}{4} D^2 = \frac{\pi}{4}(0.30)^2 = 0.07069\ m^2$$

Select a control volume as shown in the sketch. For this case of steady flow, Eq. (1.12) is applicable

$$\iint_{cs} \rho\ (\boldsymbol{V}.\hat{n})\ dA = 0 = \sum \rho\ V\ A$$

The rate of influx into the control volume is

$2 + 8 = 10.0\ kg/s$.

The rate of efflux is

$$(\rho\ V\ A)_{exit} = (\rho\ V\ A)_{in} = 10.0\ kg/s$$

$$V = \frac{10}{(0.18)(0.07069)} = 785.9\ m/s$$

Example 2.3 An air stream at a velocity of $100\ m/s$ and density of $1.2\ kg/m3$ strikes a stationary plate and is deflected by 90°. Determine the force on the plate. Assume standard atmospheric pressure surrounding the jet and an initial jet diameter of $2\ cm$.

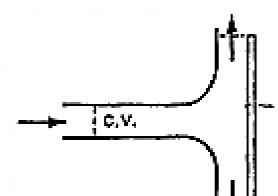


Figure 2.6a

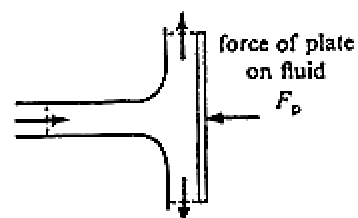


Figure 2.6b

solution

Select a control volume as shown in Figure (2.6a). Writing the x component of eq. (2.30) for steady flow to determine fluid force on the plate

$$\sum F_x = \iint_{cs} V_x\ \rho\ (\boldsymbol{V}.\hat{n})\ dA$$

$$F_{x,fluid} = 100 * \left[1.2(100)\frac{\pi}{4}(0.02)^2\right] = 3.770\ N$$

This force is opposite by F_{plate}

Example 2.4 A rocket motor is fired in place on a test stand. The rocket exhausts $10\ kg/s$ at an exit velocity of $800\ m/s$. Assume uniform steady conditions at the exit plane with an exit plane static pressure of $50\ kPa$. For an ambient pressure of $101\ kPa$, determine the rocket motor thrust transmitted to the test stand as shown in Figure (2.7).

Solution

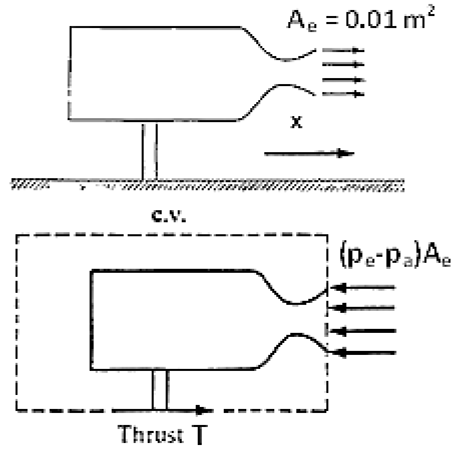


Figure 2.7

$$\sum F_x = \iint_{cs} V_x\, \rho\, (\boldsymbol{V}.\hat{n})\, dA$$

$$\sum F_x = F_{thrust} + F_{pressure}$$

$$\iint_{cs} V_x\, \rho\, (\boldsymbol{V}.\hat{n})\, dA = V_x\, \rho\, V_x\, A = \dot{m}_x V_x$$

$$F_{thrust} - (p_e - p_a)A_e = \dot{m}_x V_x$$

$$F_{thrust} = (50 - 101) \times 10^3 * 0.01 + 10 * 800$$

$$= -510 + 8000 = 7490\ N$$

Example 2.5 A rigid, well-insulated vessel is initially evacuated. A valve is opened in a pipeline connected to the vessel, which allows air at 3 MPa and 300 K to flow into the vessel. The valve is closed when the pressure in the vessel reaches 3 MPa. Determine the final equilibrium temperature of the air in the vessel over the temperature range of interest.

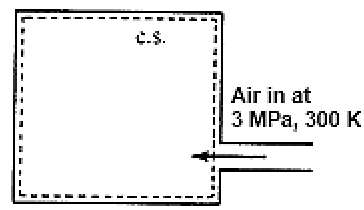


Figure 2.8

Solution

Select a control volume as shown in Figure (1.9). With no heat transfer, no work, and negligible ΔkE and ΔpE, the energy equation is

$$\left[Q + \left[\dot{m}\left(h + \frac{V^2}{2} + gz\right)\right]_{in}\right] - \left[W_s + \left[\dot{m}\left(h + \frac{V^2}{2} + gz\right)\right]_{out}\right] = (\dot{m}u)_2 - (\dot{m}u)_1$$

$$\dot{m}_{out} - \dot{m}_{in} = \dot{m}_2 - \dot{m}_1$$

$$\dot{m}_{in} = \dot{m}_2 = m$$

$$\dot{m}_{out} = \dot{m}_1 = 0$$

So eq. (1.32) is simplify to

$$(\dot{m}h)_{in} = \quad (\dot{m}u)_2$$

and

$$c_p T_{in} = c_v T_2$$

$$T_{final} = T_2 = \frac{c_p}{c_v} T_{in} = \frac{1.005}{0.718} * 300 = 421.1\ \text{K}$$

Example 2.6 Steam enters an ejector (Figure 2.9) at the rate of $0.0454\,kg/sec$ with an enthalpy of $3023.8\,kJ/kg$ and negligible velocity. Water enters at the rate of $0.454\,kg/sec$ with an enthalpy of $93\,kJ/kg$ and negligible velocity. The mixture leaves the ejector with an enthalpy of $349\,kJ/kg$ and a velocity of $27.432\,m/s$. All potentials may be neglected. Determine the magnitude and direction of the heat transfer.

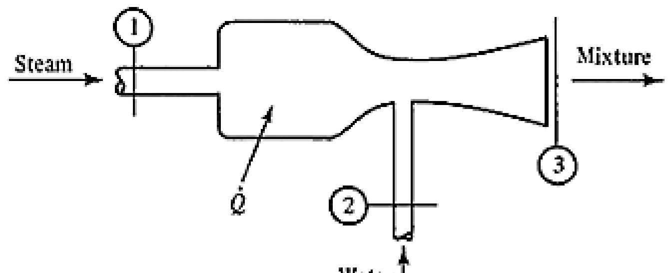


Figure 2.9

$\dot{m}_1 = 0.0454\,kg/\sec$, $\dot{m}_2 = 0.454\,kg/\sec$,
$h_1 = 3023.8\,kJ/kg$, $h_2 = 93\,kJ/kg$, $h_3 = 349\,kJ/kg$
$V_1 \approx 0.0\,m/s$, $V_2 \approx 0.0\,m/s$, $V_3 = 27.432\,m/s$
$\dot{m}_3 = \dot{m}_1 + \dot{m}_2 = 0.0454 + 0.454 = 0.4994\ kg/sec$

$$\dot{Q} + \dot{m}_1\left(h_1 + \frac{V_1^2}{2} + gz_1\right) + \dot{m}_2\left(h_2 + \frac{V_2^2}{2} + gz_2\right) = \dot{W}_s + \dot{m}_3\left(h_3 + \frac{V_3^2}{2} + gz_3\right)$$

$$\dot{Q} + \dot{m}_1 h_1 + \dot{m}_2 h_2 = \dot{W}_s + \dot{m}_3\left(h_2 + \frac{V_3^2}{2}\right)$$

$$\dot{Q} + 0.0454 * 3023.8 + 0.454 * 93 = 0.4994\left(349 + \frac{27.432^2 * 10^{-3}}{2}\right)$$

$$\dot{Q} + 137.281 + 42.222 = 550.1$$

$$\dot{Q} = -5.0245\ kW$$

Example 2.7 A horizontal duct of constant area contains CO2 flowing isothermally (Figure 2.10). At a section where the pressure is $14\,bar$ absolute, the average velocity is know to be $50\,m/s$. Farther downstream the pressure has dropped to $7\,bar$ abs. Find the heat transfer.

Solution

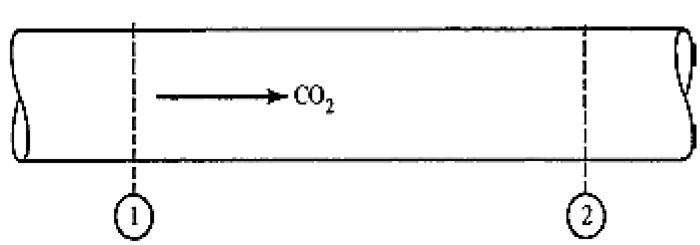


Figure 2.10

$p_1 = 14 \times 10^5\ N/m^2$
$p_7 = 7 \times 10^5\ N/m^2$
$V_1 = 50\ \ m/s$
$V_2 =?\ \ m/s$
From state equation between 1 and 2, as T is constant:

$P_1\,v_1 = p_2 v_2$

$$\frac{\rho_1}{\rho_2} = \frac{p_1}{p_2} = \frac{14}{7} = 2$$

From continuity equation

$$\dot{m} = \rho_1\,V_1\,A_1 = \rho_2\,V_2\,A_2$$

$$V_2 = V_1 * \frac{\rho_1}{\rho_2} = 50 * 2 = 100\ m/s$$

$$q = w_s + \left(u_2 + \frac{p_2}{\rho_2} + \frac{V_2^2}{2} + gz_2\right) - \left(u_1 + \frac{p_1}{\rho_1} + \frac{V_1^2}{2} + gz_1\right)$$

$$q = \left(\frac{V_2^2 - V_1^2}{2}\right) = \frac{(100^2 - 50^2)}{2} = 3750\ J/kg$$

Example 2.8 Hydrogen is expanded isentropically in a nozzle from an initial pressure of $500\ kPa$, with negligible velocity, to a final pressure of $100\ kPa$. The initial gas temperature is $500\ K$. Assume steady flow with the hydrogen behaving as a perfect gas with constant specific heats, where $c_v = 14.5\ kJ/kg.K$ and $R = 4.124\ kJ/kg.K$. Determine the final gas velocity and the mass flow through the nozzle for an exit area of $500\ m^2$.

Solution

$$\gamma = \frac{c_p}{c_v} = \frac{c_p}{c_p - R} = \frac{14.5}{14.5 - 4.124} = 1.397$$

From isentropic relation

$$T_2 = T_1 \frac{p_2}{p_1}^{\gamma-1/\gamma} = 500\left(\frac{100}{500}\right)^{1.397-1/1.397} = 316.5\ K$$

From energy equation

$$q = w_s + \left(h + \frac{V^2}{2} + gz\right)_{out} - \left(h + \frac{V^2}{2} + gz\right)_{in}$$

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

$$V_2 = \sqrt{2(h_1 - h_2)} = \sqrt{2cp(T_1 - T_2)} = \sqrt{2 * 14.5 * 10^3(500 - 316.5)} = 2306.84\ \ m/s$$

From equation of state

$$\rho_2 = \frac{p_2}{RT_2} = \frac{100}{4.124 * 316.5} = 0.0766\,kg/m^3$$

From continuity equation

$$\dot{m} = \rho_2 V_2 A_2 = 0.0766 * 2306.84 * (500 * 10^4) = 8.837\ \ kg/s$$

Example 2.9 There is a steady one-dimensional flow of air through a 30.48 cm diameter horizontal duct (Figure 1.12). At a section where the velocity is 140.208 m/s, the pressure is 344.379 kN/m^2 and the temperature is 305.5 K. At a downstream section the velocity is 268.224 m/s and the pressure is 164.7847 kN/m^2. Determine the total wall shearing force between these sections.

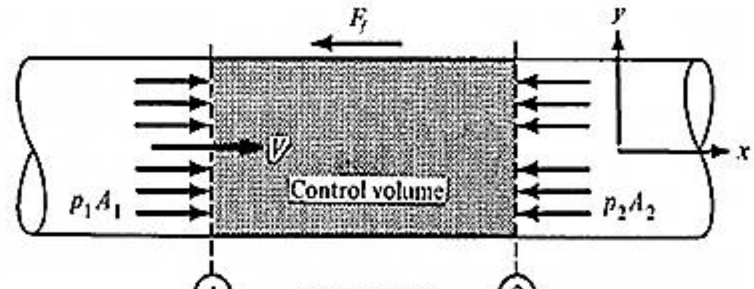


Figure 2.11

Solution

From eq.

$$\sum \mathbf{F} = \sum \dot{m}\,(\mathbf{V}_{\boldsymbol{out}} - \mathbf{V}_{\boldsymbol{in}})$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{344.379}{0.287 * 305.5}$$

$$= 3.928\ \ kg/m^3$$

$$\dot{m} = \rho_1 V_1 A = 3.928 * 140.208 * \pi * 0.3048^2/4 = 40.182\ \ kg/s$$

$$\sum \mathrm{F} = (pA)_1 - (pA)_2 - F_f$$

$$F_f = (pA)_1 - (pA)_2 + m(V_{exit} - V_{in})$$

$$F_f = (344.379 - 164.7847) * 10^3 * \frac{\pi}{4} 0.3048^2 + 40.182\ (268.224 - 140.379)$$

$$= 13104.256 - 5137.067 = 7967.2\ N$$

Chapter Three/Wave Propagation

3.1. Introduction

The method by which a flow adjusts to the presence of a body can be shown visually by a plot of the flow streamlines about the body. Figures (3.1) and (3.2) show the streamline patterns obtained for uniform, steady, incompressible flow over an airfoil and over a circular cylinder, respectively.

Note that the fluid particles are able to sense the presence of the body before actually reaching it. At points 1 and 2, for example, the fluid particles have been displaced vertically, yet 1 and 2 are points in the flow field well ahead of the body. This result, true in the general case of anybody inserted in an incompressible flow, suggests that a signaling mechanism exists whereby a fluid particle can be forewarned of a disturbance in the flow ahead of it. The velocity of signal waves sent from the body, relative to the moving fluid, apparently is greater than the absolute fluid velocity, since the flow is able to start to adjust to the presence of a body before reaching it.

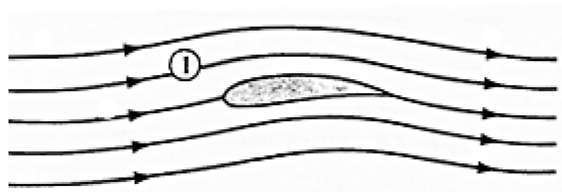


Figure 3.1

Thus, when a body is inserted into incompressible flow, a smooth, continuous streamlines result, which indicate gradual changes in fluid properties as the flow passes over the body. If the fluid particles were to move faster than the signal waves, the fluid would not be able to sense the body before actually reaching it. and very abrupt changes in velocity vectors and other properties would ensue.

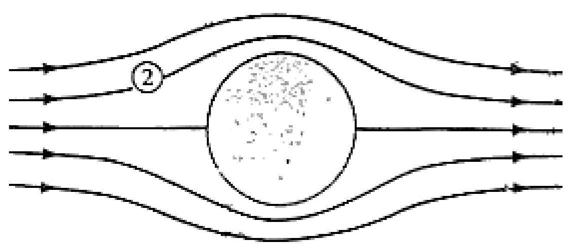


Figure 3.2 Stream patterns for steady incompressible flow

In this chapter, the mechanism by which the signal waves are propagated through incompressible and compressible flows will be studied. An expression for the velocity of propagation of the waves will be derived.

3.2. Wave formulation

To examine the means by which disturbances pass through an elastic medium. A disturbance at a given point creates a region of compressed molecules that is passed along to its neighboring molecules and in so doing creates a *traveling wave*. Waves come in various *strengths*, which are measured by the amplitude of the disturbance. The speed at which this disturbance is propagated through the medium is called the *wave speed*. This speed not only depends on the type of medium and its thermodynamic state but is also a function of the strength of the wave. The *stronger* the wave is, the faster it moves.

If we are dealing with waves of *large amplitude*, which involve relatively large changes in pressure and density, we call these *shock waves*. These will be studied later. If, on the other hand, we observe waves of *very small amplitude*, their speed is characteristic only by the medium and its state. These waves are of vital importance since sound waves fall into this category. Furthermore, the presence of an object in a medium can only be felt by the object's sending out or reflecting infinitesimal waves which propagate at the *sonic velocity*.

Consider a long constant-area tube filled with fluid and having a piston at one end, as shown in Figure (3.3). The fluid is initially at rest. At a certain instant the piston is given an incremental velocity dV to the left. The fluid particles immediately next to the piston are compressed a very small amount as they acquire the velocity of the piston. As the piston (and these compressed particles) continue to move, the next group of fluid particles is compressed and the *wave front* is observed to propagate through the fluid at *sonic velocity* of magnitude a. All particles between the wave front and the piston are moving with velocity dV to the left and have been compressed from ρ to $\rho + d\rho$ and have increased their pressure from p to $p + dp$.

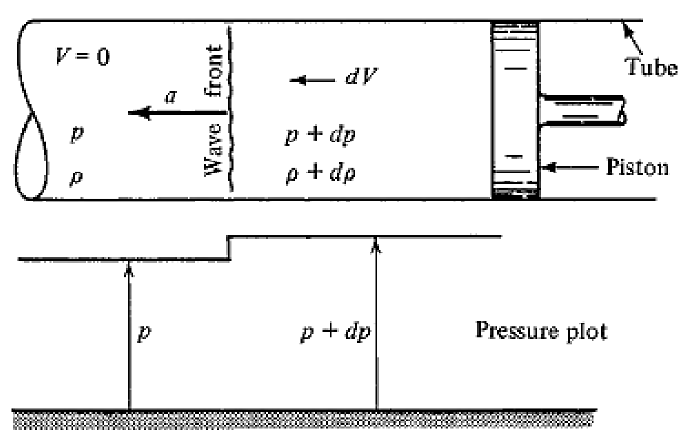


Figure 3.3 Initiation of infinitesimal pressure pulse.

The flow is unsteady and the analysis is difficult. This difficulty can easily be solved by superimposing on the entire flow field a constant velocity to the right of magnitude a.

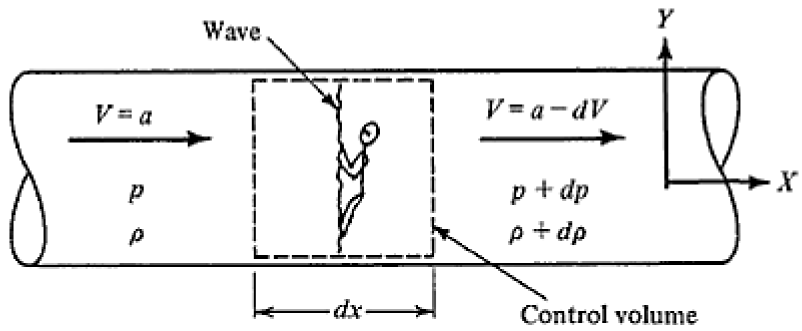


Figure 3.4 Steady-flow picture corresponding to Figure 3.3.

3.3. Sonic Velocity

Figure (3.4) shows the problem. Since the wave front is extremely thin, we can use a control volume of infinitesimal thickness. For steady one-dimensional flow, we have from continuity equation

$$\dot{m} = \rho A V = const$$

But $A = const$; thus

$$\rho V = const \tag{3.1}$$

Application of this to our problem yields

$$\rho a = (\rho + d\rho)(a - dV)$$

$$\rho a = \rho a - \rho\, dV + a\, d\rho - d\rho\, dV$$

Neglecting the higher-order term and solving for dV, we have

$$dV = \frac{a\, d\rho}{\rho} \tag{3.2}$$

Since the control volume has infinitesimal thickness, we can neglect any shear stresses along the walls. We shall write the x-component of the momentum equation, taking forces and velocity as positive if to the right. For steady one-dimensional flow we may write from momentum equation

$$\sum \mathrm{F}_x = \sum \dot{m}\,(\mathrm{V}_{out} - \mathrm{V}_{in})$$

$$pA - (p + dp)A = \rho A a[(a - dV) - a]$$

$$A dp = \rho A a\, dV$$

Canceling the area and solving for dV, we have

$$dV = \frac{dp}{\rho a} \tag{3.3}$$

Equations (3.2) and (3.3) may now be combined, the result is:

$$a^2 = \frac{dp}{d\rho} \qquad (3.4a)$$

However, the derivative $dp/d\rho$ is not unique. It depends entirely on the process. For example

$$\left(\frac{\partial p}{\partial \rho}\right)_T \neq \left(\frac{\partial p}{\partial \rho}\right)_s$$

Thus it should really be written as a *partial* derivative with the appropriate subscript.

Since we are analyzing an infinitesimal disturbance, we can assume negligible losses and heat transfer as the wave passes through the fluid. Thus the process is both reversible and adiabatic, which means that it is isentropic. Equation (4.4) should properly be written as:

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_{ise} \qquad (3.4b)$$

For substances other than gases, sonic velocity can be expressed in an alternative form by introducing the *bulk* or *volume modulus of elasticity Ev*.

$$E_v = -v\left(\frac{\partial p}{\partial v}\right)_{ise} \equiv \rho\left(\frac{\partial p}{\partial \rho}\right)_{ise} \qquad (3.5)$$

$$a^2 = \frac{E_v}{\rho} \qquad (3.6)$$

Equations (3.4) and (3.6) are equivalent general relations for sonic velocity through *any* medium. The bulk modulus is normally used in connection with liquids and solids. Table 4.1 gives some typical values of this modulus, the exact value depending on the temperature and pressure of the medium. For solids it also depends on the type of loading. The reciprocal of the bulk modulus is called the *compressibility*.

Table 4.1 Bulk Modulus Values for Common Media

Medium	Bulk Modulus (psi)
Oil	185,000–270,000
Water	300,000–400,000
Mercury	approx. 4,000,000
Steel	approx. 30,000,000

Equation (3.4) is normally used for gases and this can be greatly simplified for the case of a gas that obeys the perfect gas law. For an isentropic process:

$$pv^{\gamma} = c \quad or \quad p = c\,\rho^{\gamma}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_{ise} = c\,\gamma\,\rho^{\gamma-1} = \gamma\,\rho^{\gamma-1}\,\frac{p}{\rho^{\gamma}} = \gamma RT$$

$$a = \sqrt{\gamma RT} \tag{3.7}$$

For perfect gases, sonic velocity is a function of the $\gamma\ , R\ and\ T\ only.$

$$Mach\ number, M = \frac{V}{a} \tag{3.8}$$

It is important to realize that both V and a are computed *locally* for the same point. For other point within the flow we must seek further information to compute on the sonic velocity, which has probably changed.

Subsonic flow, $M <$, the velocity is less than the local speed of sound.

Supersonic flow, $M > 1$, the velocity is greater than the local speed of sound.

We shall soon see that the Mach number is the most important parameter in the analysis of compressible lows.

3.4: Wave Propagation

Let us examine a point disturbance that is at rest in a fluid. *Infinitesimal* pressure pulses are continually being emitted and thus they travel through the medium at *sonic* velocity in the form of spherical wave fronts. To simplify matters we shall keep track of only those pulses that are emitted every second. At the end of 3 seconds the picture will appear as shown in Figure (3.5). Note that the wave fronts are concentric.

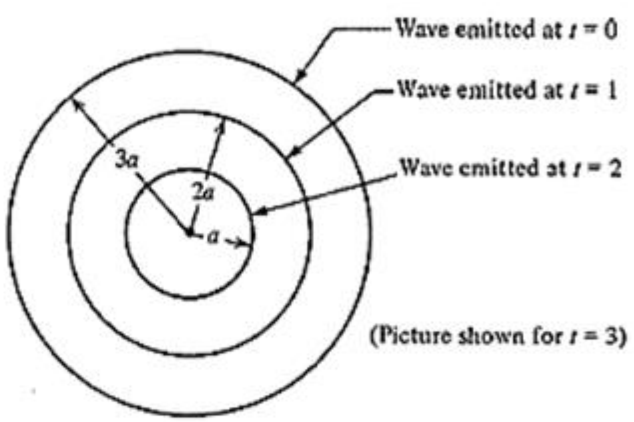


Figure 3.5 Wave fronts from a stationary disturbance.

Now consider a similar problem in which the disturbance is moving at a speed less than sonic velocity, say $a/2$. Figure (3.6) shows such a situation at the end of 3 seconds. Note that the wave fronts are no longer concentric.

Furthermore, the wave that was emitted at $t = 0$ is always in front of the disturbance itself. *Therefore, any person, object, or fluid particle located upstream will feel the wave fronts pass by and know that the disturbance is coming*.

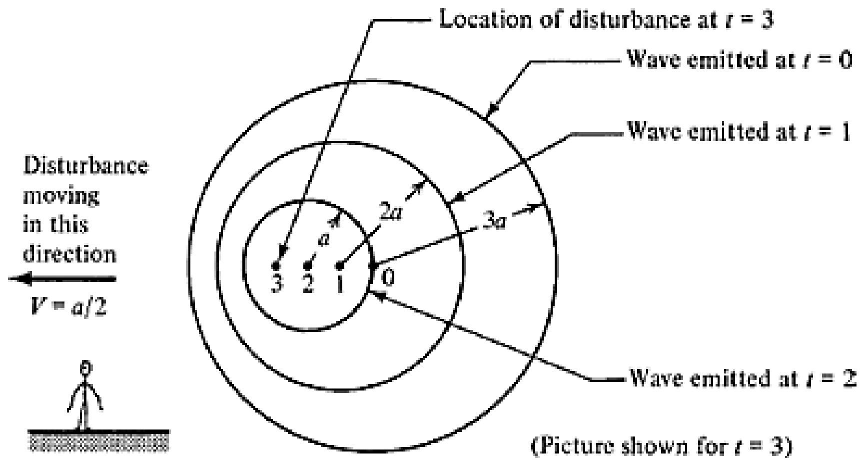
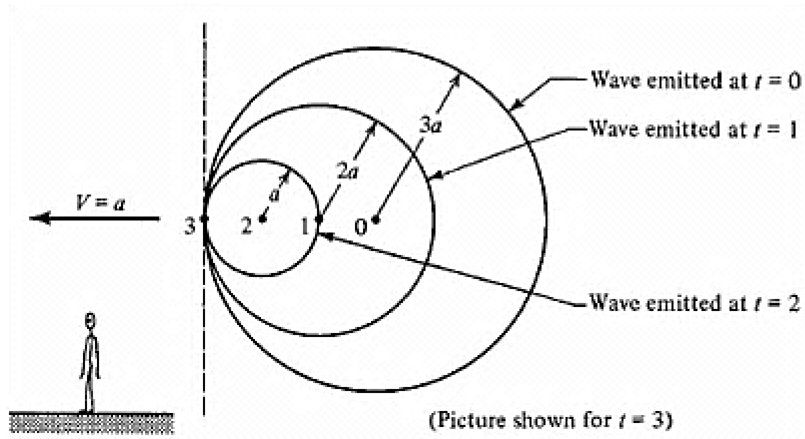


Figure 3.6 Wave fronts from subsonic disturbance.

Next, let the disturbance move at exactly sonic velocity. Figure (3.7) shows this case and you will note that all wave fronts coalesce on the left side and move along with the disturbance. After a long period of time this wave front would approximate a plane indicated by the dashed line. In this case, no region upstream is forewarned of the disturbance as the disturbance arrives at the same time as the wave front.



(Picture shown for $t = 3$)

The only other case to consider is that of a disturbance moving at velocities greater than the speed of sound. Figure (3.8) shows a point disturbance moving at Mach number = 2 (twice sonic velocity). The wave

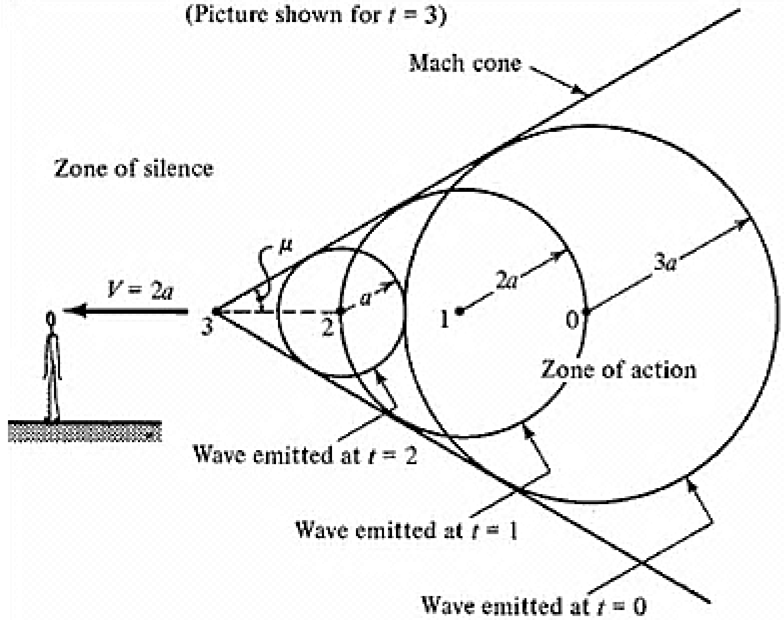


Figure 3.8 Wave fronts from supersonic disturbance.

fronts have coalesced to form a cone with the disturbance at the apex. This is called a *Mach cone*. The region inside the cone is called the *zone of action* since it feels the presence of the waves. The outer region is called the *zone of silence*, as *this entire region is unaware of the disturbance*. The surface of the Mach cone is sometimes referred to as a *Mach wave*; the half-angle at the apex is called the *Mach angle* and is given the symbol μ. It should be easy to see that:

$$\sin\mu = \frac{a}{V} = \frac{1}{M} \tag{3.9}$$

For subsonic flow, no such zone of silence exists. If the disturbance caused by a projectile, the entire fluid is able to sense the projectile moving through it, since the signal waves move faster than the projectile. No concentration of pressure disturbances can occur for subsonic flow; Mach lines cannot be defined.

Let us now compare steady, uniform, subsonic and supersonic flow over a finite wedge-shaped body. If the fluid velocity is less than the velocity of sound, flow ahead of the body is able to sense its presence. As a result, gradual changes in flow properties take place; with smooth, continuous streamlines (see Figure 3.9).

If the fluid velocity is greater than the velocity of sound, the approach flow, being in the zone of silence, is unable to sense the presence of the body. The body now presents a finite disturbance to the flow. The wave pattern obtained is a result of the addition of individual Mach waves emitted from each point on the wedge. This nonlinear addition yields a compression shock wave across which occur finite changes in velocity, pressure, and other flow properties. A typical flow pattern obtained for supersonic flow over the wedge is shown in Figure (3.10).

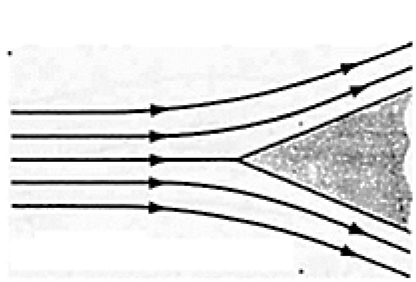


Figure 3.9 Subsonic wedge Flow

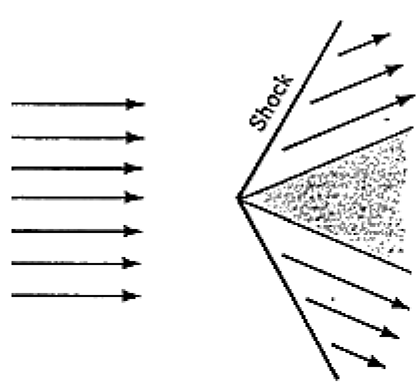


Figure 3.10 Supersonic wedge flow

Chapter Four/Isentropic flow of a perfect gas in varying area duct

To study the compressible, isentropic flow through varying area channels such as nozzles, diffusers and turbine blade passages, the following assumptions are considered:

1. One dimensional, steady flow of a perfect gas.
2. Friction is zero.
3. No heat and work exchange.
4. Variation in properties is brought about by area change.
5. Changes in potential energy and gravitational forces are negligible.

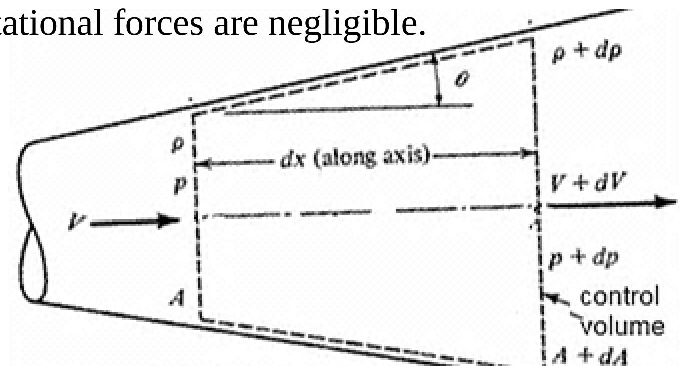


Figure 4.1 Control volume for varying area flow

4.1 Equations of motion.

- **Continuity equation:**

$$\iint_{cs} \rho\,(\boldsymbol{V}.\hat{n})\,dA = \sum \rho\,V\,A = 0 \qquad (4.1)$$

$$\dot{m} = \rho\,V\,A = const \qquad (4.2)$$

$$(\rho + d\rho)(\,V + dV)(\,A + dA) = \rho\,V\,A \qquad (4.3)$$

Simplifying and ignoring high order

$$\rho VA + \rho V dA + \rho A dV + VA d\rho = \rho\,V\,A \qquad (4.4)$$

Divided by $\rho\,V\,A$

$$\boldsymbol{\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0} \qquad (4.5)$$

- **Momentum equation:**

$$\sum \mathbf{F} = \iint_{cs} \mathbf{V}\,\rho\,(\boldsymbol{V}.\hat{n})\,dA \qquad (4.6)$$

$$\iint_{cs} \mathbf{V}\,\rho\,(\boldsymbol{V}.\hat{n})\,dA = \rho VA[(V + dV) - V] \qquad (4.7)$$

If there is no electromagnetic force and friction force is negligible, the only acting force is the pressure force. The side wall pressure force in flow direction can be obtained with a mean pressure value:

wall pressure force $= [(mean\ pressure)(wall\ area)]\ sin\ \theta$

but $dA = (wall\ area)\ sin\ \theta$; and thus

$$\text{wall pressure force } = \left(p + \frac{dp}{2}\right) dA \tag{4.8}$$

$$\sum \mathbf{F} = pA + \left(p + \frac{dp}{2}\right) dA - (p + dp)(A + dA) \tag{4.9}$$

$$pA + \left(p + \frac{dp}{2}\right) dA - (p + dp)(A + dA) = \rho VA[(V + dV) - V] \tag{4.10}$$

Simplifying and ignoring high orders

$$\boldsymbol{dp + \rho V dV = 0} \tag{4.11}$$

- **Energy equation**

$$\iint_{cs} e\ \rho\ (\boldsymbol{V}.\hat{n})\ dA = 0 \tag{4.12}$$

$$\iint_{cs} [\delta q - \delta w_s + d(u + pv + k.e. + p.e)]\ \rho\ (\boldsymbol{V}.\hat{n})\ dA = 0 \tag{4.13}$$

The specific energy e is stand for internal, flow, kinetic and potential energies, since there is no heat and work transfer. Then from S.F.E.E.;

$$\delta q + \left(pv + u + \frac{V^2}{2} + gz\right) = \delta w_s + \left((p + dp)(v + dv) + (u + du) + \frac{(V + dV)^2}{2} + g(z + dz)\right)$$

$$0 = \left(pdv + vdp + du + \frac{2VdV}{2}\right) \tag{4.14}$$

$$0 = dh + \frac{dV^2}{2} \tag{4.15}$$

Substitute from thermodynamics relations

$$\delta q = dW_s + du = pdv + du = dh - vdp = 0$$

$$dh = vdp$$

$$\boldsymbol{dp + \rho V dV = 0} \qquad (4.16)$$

This is the energy equation which is similar to equation (4.11).

4.2 Stagnation concept and relations

If you had a thermometer and pressure gage, they would indicate the temperature and pressure corresponding to the *static* state of the fluid, as you move with flow velocity. Thus *the static properties are those that would be measured if you moved with the fluid*.

Stagnation state defined as that thermodynamic state which would exist if the fluid were brought to zero velocity and zero potential. To yield a consistent reference state, we must qualify how this *stagnation process* should be accomplished. The stagnation state must be reached

1. Without any energy exchange ($Q = W = 0$)
2. Without friction losses.

From (1), change of entropy due to energy exchange is zero, i.e. $ds_{ext} = 0$; and from (2), change of entropy due to friction is zero, i.e. $ds_{int} = 0$. Thus *the stagnation process is isentropic*!

Consider fluid that is flowing and has the static properties shown as (a) in Figure 4.3. At location (b) the fluid has been brought to zero velocity and zero potential under the foregoing restrictions. If we apply the energy equation to the control volume indicated for steady one-dimensional flow, we have.

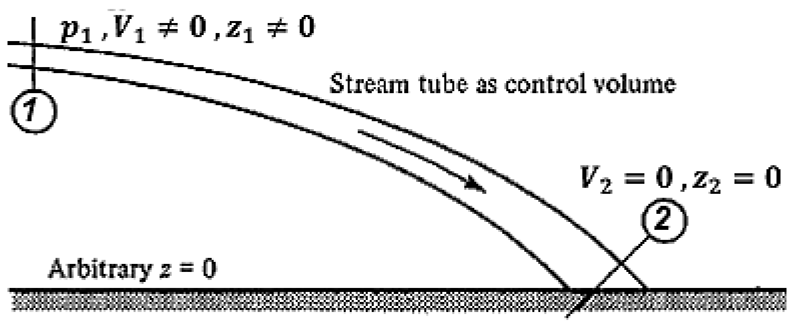


Figure 3.1 Stagnation Process

$$q + \left(h_1 + \frac{V_1^2}{2} + gz_1\right) = w_s + \left(h_2 + \frac{V_2^2}{2} + gz_2\right)$$

$$h_1 + \frac{V_1^2}{2} + gz_1 = h_2 \qquad (4.17)$$

Since condition (2) represents the *stagnation state* corresponding to the *static state* (1). Thus we call h_2 the *stagnation* or *total enthalpy* corresponding to state (1) and designate it as h_{to}. Thus

$$h_{to} = h_1 + \frac{V_1^2}{2} + gz_1$$

Or for any state, we have in general,

$$h_o = h + \frac{V^2}{2} + gz \qquad (4.18)$$

This is an important relation that is *always* valid. When dealing with gases, potential energy changes are usually neglected, and we write.

$$h_o = h + \frac{V^2}{2} \qquad (4.19)$$

The one-dimension S.F.E.E. becomes:

$$h_{o1} + q = h_{o2} + w_s \qquad (4.20a)$$

$$h_{o1} = h_{o2} \quad or \quad dh_o = 0 \qquad (4.20b)$$

Equation (4.20) shows that for any adiabatic, no-work, steady, one-dimensional flow system, the stagnation enthalpy remains constant, *irrespective of the losses*.

One must realize that when the frame of reference is changed, stagnation conditions change, although the static conditions remain the same. Consider still air with Earth as a reference frame. In this case, since the velocity is zero the static and stagnation conditions are the same. For gases we eliminate potential term

$$c_p = \frac{\gamma R}{\gamma - 1}, \qquad h = c_p T$$

$$h_o = h + \frac{V^2}{2} = h + \frac{M^2 \gamma R T}{2} = h + M^2 \frac{\gamma - 1}{2} c_p T$$

$$h_o = h \left(1 + M^2 \frac{\gamma - 1}{2}\right) \qquad (4.21)$$

$$T_o = T \left(1 + M^2 \frac{\gamma - 1}{2}\right) \qquad (4.22)$$

The stagnation process is isentropic. Thus γ is used as the exponent in the relations between any two points on the same isentropic streamline. Let point 1 refers to the static conditions, and point 2, the stagnation conditions. Then,

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\gamma/(\gamma-1)}$$

$$p_o = p\left(1 + M^2\frac{\gamma-1}{2}\right)^{\gamma/(\gamma-1)} \qquad (4.23)$$

$$\rho_o = \rho\left(1 + M^2\frac{\gamma-1}{2}\right)^{1/(\gamma-1)} \qquad (4.24)$$

Example 4.1 Air flows with a velocity of $243.84\ m/s$ and has a pressure of $206.843\ kN/m^2$ and temperature of $60.2\ °C$. Determine the stagnation pressure.

Solution

$$a = \sqrt{\gamma RT} = \sqrt{1.4 * 287 * (60.2 + 273)} = 365.9\ m/s$$

$$M = \frac{V}{a} = \frac{243.84}{365.9} = 0.666$$

$$p_o = p\left(1 + M^2\frac{\gamma-1}{2}\right)^{\gamma/(\gamma-1)} = 206.843\left(1 + 0.666^2\frac{1.4-1}{2}\right)^{(1.4/1.4-1)}$$
$$= 278.506\ kN/m^2$$

Example 4.2 Hydrogen, $\gamma_{Hy} = 1.405$, has a static temperature of $25°C$ and a stagnation temperature of $250°C$. What is the Mach number?

Solution

$$T_o = T\left(1 + M^2\frac{\gamma-1}{2}\right)$$

$$(250 + 273) = (25 + 273)\left(1 + M^2\frac{1.405-1}{2}\right)$$

$$523 = 293\,(1 + 0.2025\,M^2) \ \rightarrow\ M^2 = 3.8765 \rightarrow M = 1.969$$

Chapter Five/Subsonic and Supersonic Flow through a Varying Area Channels

5.1 Isentropic Flow in varying Area ducts

For isentropic flow, from continuity

$$\frac{d\rho}{\rho}+\frac{dA}{A}+\frac{dV}{V}=0 \qquad (4.5)$$

and from momentum equations

$$dp+\rho V dV=0 \qquad (4.11)$$

$$dV=-\frac{dp}{\rho V}$$

Substitute into momentum eq.

$$\frac{d\rho}{\rho}+\frac{dA}{A}-\frac{dp}{\rho V^2}=0 \qquad (5.1a)$$

$$dp-\rho V^2\left(\frac{d\rho}{\rho}+\frac{dA}{A}\right)=0 \qquad (5.1b)$$

From definition of sonic velocity, eq.3.4

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_{ise}=\left(\frac{dp}{d\rho}\right)_{ise} \Longrightarrow d\rho = \frac{dp}{a^2}$$

$$dp-\rho V^2\left(\frac{dp}{\rho a^2}+\frac{dA}{A}\right)=0$$

$$dp-M^2dp=\rho V^2\frac{dA}{A}$$

$$dp=\rho V^2\left(\frac{1}{(1-M^2)}\right)\frac{dA}{A} \qquad (5.2a)$$

$$p=\rho RT=\frac{\rho}{\gamma}a^2$$

$$\frac{dp}{p} = \left(\frac{\gamma M^2}{(1 - M^2)}\right)\frac{dA}{A} \qquad (5.2b)$$

Also from eq. 5.1. after substitute for $dp = a^2 dp$ from definition of sonic velocity

$$\frac{d\rho}{\rho} + \frac{dA}{A} - \frac{dp}{\rho V^2} = 0$$

$$\frac{d\rho}{\rho} + \frac{dA}{A} - \frac{1}{M^2}\frac{d\rho}{\rho}$$

$$\frac{d\rho}{\rho} = \frac{M^2}{(1 - M^2)}\left(\frac{dA}{A}\right) \qquad (5.3)$$

Substitute eq.5.3 into continuity eq.4.5. gives

$$\frac{M^2}{(1 - M^2)}\frac{dA}{A} + \frac{dA}{A} + \frac{dV}{V} = 0$$

$$\frac{dV}{V} = -\left(\frac{1}{1 - M^2}\right)\left(\frac{dA}{A}\right) \qquad (5.4)$$

Let us consider what is happening to fluid properties as it flows through a variable-area duct.
For subsonic flow, $M < 1$, then $(1 - M^2)\ is + ve$.

When dA is negative (area is decreasing), then dp is negative (pressure decreases) and $d\rho$ is negative (density decreases) and dV is positive (velocity increases) and vice versa.

For supersonic flow, $M > 1$, then $(1 - M^2)\ is - ve$.

When dA is negative (area is decreasing), then dp is positive (pressure increases) and $d\rho$ is positive (density increases) and dV is negative (velocity decreases) and vice versa.

We summarize the above by saying that *as the pressure decreases*, the following variations occur:

		Subsonic ($M < 1$)	Supersonic ($M > 1$)
Area	A	Decreases	Increases
Density	ρ	Decreases	Decreases
Velocity	V	Increases	Increases

Table 5.1: Variation of area, density and velocity with Much number as the pressure decreases

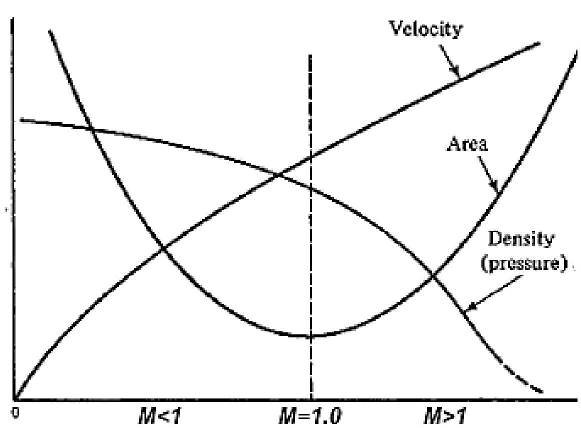


Figure 5.1: Property variation with Mach number

Combines equations (5.4) and (5.3) to eliminate the term dA/A with the following result:

$$\frac{d\rho}{\rho} = -M^2\left(\frac{dV}{V}\right) \tag{5.5}$$

From this equation we see that:

At **low Mach** numbers, density variations will be quite small. This means that the density is nearly constant ($d\rho = 0$) in the low subsonic regime ($M \leq 0.3$) and the velocity changes compensate for area changes.

At a **Mach** number equal to **unity**, we reach a situation where density changes and velocity changes compensate for one another and thus no change in area is required ($dA = 0$).

At **supersonic** flow, the density decreases so rapidly that the accompanying velocity change cannot accommodate the flow and thus the area must increase.

A ***nozzle*** is a device that converts enthalpy (or pressure energy for the case of an incompressible fluid) into kinetic energy. From Figure 5.1 we see that an increase in velocity is accompanied by either an increase or decrease in area, depending on the Mach number. Figure 5.2 shows what these devices look like in the subsonic and supersonic flow regimes.

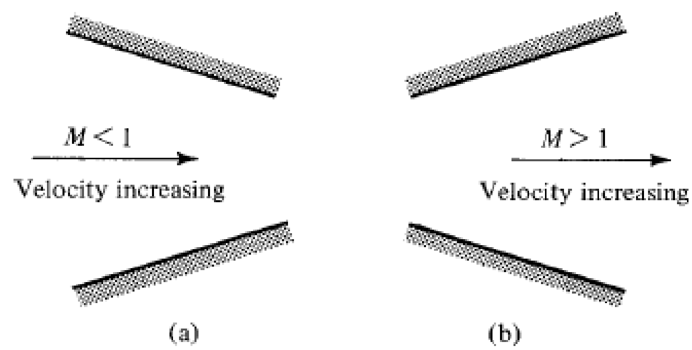


Figure 5.2 Nozzle configurations.

A ***diffuser*** is a device that converts kinetic energy into enthalpy (or pressure energy for the case of incompressible fluids). Figure 5.3 shows what these devices look like in the subsonic and supersonic regimes. Thus we see that the same piece of equipment can operate as either a nozzle or a diffuser, depending on the flow regime.

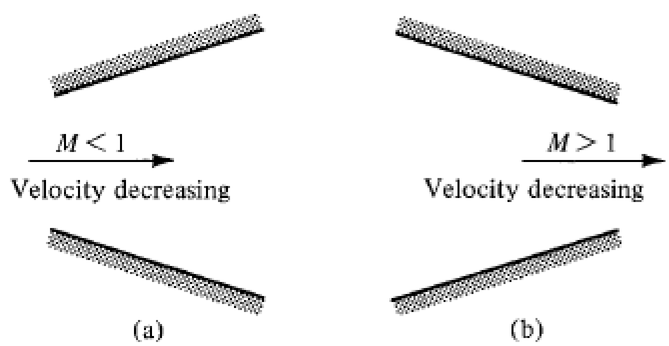


Figure 5.3 Diffuser configurations.

Notice that a device is called a nozzle or a diffuser because of *what it does*, not what it looks like.

Further consideration of Figures 5.1 and 5.2 leads to some interesting conclusions. If one attached a converging section (see Figure 5.2*a*) to a high-pressure supply, one could never attain a flow greater than Mach 1, regardless of the pressure difference available. On the other hand, if we made a converging–diverging device (combination of Figure 5.2*a* and *b*), we see a means of accelerating the fluid into the supersonic regime, provided that the proper pressure difference exists between inlet and exit plane.

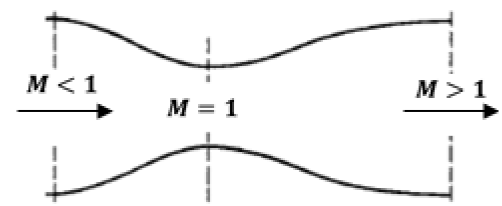


Figure 5.4: Convergent-divergent nozzle

5.2 The (∗) Reference Concept

Concept of a stagnation reference state was introduced which is an *isentropic process*. It will be convenient to introduce another reference condition since the stagnation state is not a feasible reference when dealing with area changes. (Why?)

The new reference state with a superscript (∗) and define it as "that thermodynamic state which would exist if the fluid reached a Mach number of unity *by some particular process*". There are many processes by which we could reach Mach 1.0 from any given starting point, and they would each lead to a different thermodynamic state.

For isentropic flow process, adiabatic frictionless, flow the stagnation properties for all points are the same as well as the (*) properties are the same.

For actual flow process, each point in the flow has its own stagnation and (*) properties.

Consider a steady, one-dimensional flow of a perfect gas with no heat or work transfer and negligible potential changes but with friction. Figure 5.5 shows a T –s diagram indicating two points in such a flow system. Above each point is shown its stagnation reference state, and below its reference state (∗).

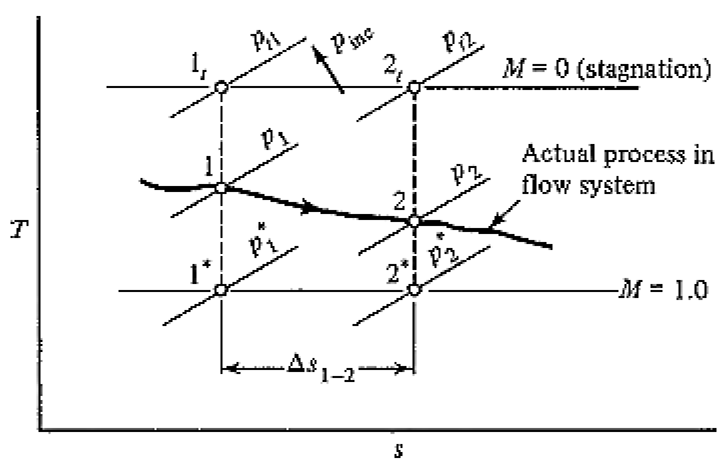


Figure 5.5 Isentropic * reference states.

Note that the stagnation temperatures are the same and lie on a horizontal line, but the stagnation pressures are different, and also ($*$) reference points will lie on another horizontal line (since no heat is added).

Between ($*$) reference state and the stagnation reference state lie all points in the subsonic regime. Below the ($*$) reference state lie all points in the supersonic regime.

5.3 Isentropic Table

Mass flow rate at flow cross sectional area A can be expressed in terms of stagnation pressure and temperature

$\dot{m} = \rho A\, V = const$ $\quad$ *continuty equation*

$p = \rho R T$ $\quad$ *state equation*

$a = \sqrt{\gamma R T}$ $\quad$ *sonic speed*

$M = V/a$ $\quad$ *Much number*

For perfect gas with constant specific heat

$$\dot{m} = \frac{p}{RT} A M \sqrt{\gamma R T} = \frac{p}{R\sqrt{T}} A M \sqrt{\gamma R} \tag{5.6}$$

Substitute for p and T from

$$T_o = T\left(1 + M^2 \frac{\gamma - 1}{2}\right) \tag{4.26}$$

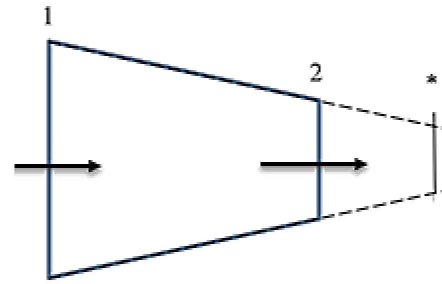
$$p_o = p\left(1 + M^2 \frac{\gamma - 1}{2}\right)^{\gamma/(\gamma-1)} \tag{4.28}$$

$$\dot{m} = \frac{p_o}{R\sqrt{T_o}} A M \sqrt{\gamma R}\left(1 + M^2 \frac{\gamma - 1}{2}\right)^{-(\gamma+1)/2(\gamma-1)} \tag{5.7}$$

$$\dot{m} = \frac{p_o A}{R\sqrt{T_o}} f(\gamma, M) \tag{5.8}$$

$$f(\gamma, M) = \frac{M\sqrt{\gamma}}{\left(1 + M^2 \frac{\gamma - 1}{2}\right)^{(\gamma+1)/2(\gamma-1)}} \tag{5.9}$$

For isentropic flow where p_o and T_o are constant, cross section A can be related directly to Mach number. Select flow cross section area where $M = 1$ as a reference area A^* . For steady flow, the mass flow rate at area A is equal to the mass flow rate at area A^*.



$$\dot{m} = \dot{m}^*$$

$$\frac{p_o A}{R\sqrt{T_o}} f(\gamma, M) = \frac{p_o A^*}{R\sqrt{T_o}} f(\gamma) \qquad (5.10)$$

$$\frac{A}{A^*} = g(\gamma, M)$$

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1 + [(\gamma - 1)/2]M^2}{(\gamma + 1)/2}\right)^{(\gamma+1)/2(\gamma-1)} \qquad (5.11)$$

The result of equation (5.11) is plotted in figure (5.6) for $\gamma = 1.4$. For each value of A/A^* there are two possible isentropic solution, one subsonic and the other supersonic. The minimum area or throat area occurs at $M = 1$. This agree well with the result of eq 5.6 that illustrated in figure 5.2. and 5.3.

A convergent-divergent nozzle is required to accelerate a slowly moving stream to supersonic velocities.

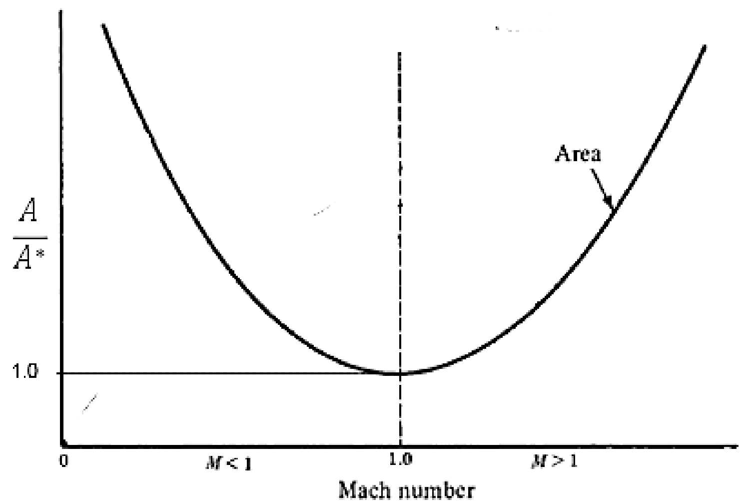


Figure 5.6 Area ratio variation with Mach number

Example: 5.1

An airstream flows in a converging duct from cross section area A_1 of $50\ cm^2$ to a cross-sectional area A_2 of $40\ cm^2$. If $T_1 = 300\ K$, $p_1 = 100\ kPa$ and $V_1 = 100\ m/s$. Find M_2, p_2 and T_2 . Assume steady one-dimensional isentropic flow.

Solution:

Over the temperature range, air behaves as perfect gas with $\gamma = 1.4$.

$$M_1 = \frac{V_1}{a} = \frac{V_1}{\sqrt{\gamma RT}} = \frac{100}{\sqrt{1.4 * 0.287 * 300}} = 0.288$$

At $M_1 = 0.288$ from isentropic flow table with $\gamma = 1.4$

$$\frac{A_1}{A^*} = 2.11$$

But

$$\frac{A_2}{A_1} = \frac{40}{50} = 0.80$$

So that

$$\frac{A_2}{A^*} = \frac{A_1}{A^*} * \frac{A_2}{A_1} = 1.689$$

From isentropic flow table , $M_2 = 0.372$

For isentropic flow, (no shaft work, potential energy is neglected for a gas), $p_t \ and \ T_t$ are constant. At $M = 0.288$ from isentropic flow table :

$$\frac{p_1}{p_{o1}} = 0.944 \rightarrow p_{t1} = \frac{100}{0.944} = 105.9 \ kPa = p_{t1}$$

$$\frac{T_1}{T_{o1}} = 0.984 \rightarrow T_{t1} = \frac{300}{0.984} = 304.9 \ K$$

At $M_2 = 0.372$

$$\frac{p_2}{p_{o1}} = 0.909 \rightarrow p_2 = 0.909 * 105.9 = 96.3 \ \ kPa$$

$$\frac{T_2}{T_{o1}} = 0.973 \rightarrow T_2 = 0.973 * 304.9 = 296.7 \ K$$