
بسم الله الرحمن الرحيم

Ministry of Higher Education and Scientific Research
Foundation of Technical Education
Technical College / Al-Najaf

Training package in
Fourier Series
For students of second class

Lecturer
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1/ Over view

1 / A –Target population :-

For students of second class in

Communications Techniques Engineering Department

1 / B –Rationale :-

In this package, we succeeded in representing periodic signals as a sum (everlasting) sinusoids or exponentials.

1 / C –Central Idea :-

1. Definition
2. Trigonometric Fourier series
3. Exponential Fourier series

1 / D –Objectives:-

- Fourier series used to analyze periodic signals
- The harmonic contant of the signals is analyzed with the help of the Fourier series.
- Fourier series can be developed for continuous time as well as discrete time signals.

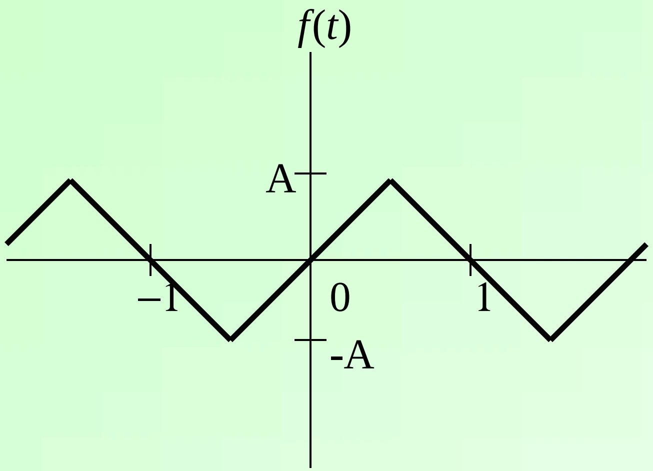
2/ Pre test :-

1. What are the Dirichlet conditions

(i) The function f(t) is single valued within the interval T0.
(ii) The function f(t) has a finite number of discontinuities
and a finite number of maxima and minima in the
interval
T0.
(iii) The function f(t) is absolutely integrable, that is,

$$\int_{-T_0}^{T_0} |f(t)|\,dt < \sim$$

2.Find the Fourier series coefficients of



$$f\left(t\right)=a_0+\sum_{n=1}^{\infty}a_n\cos(\pi\, n\, t)+b_n\sin\left(\pi\, n\, t\right)$$

$$a_0=0 \quad \text{(by inspection of the plot)}$$

$$a_n=0 \quad \text{(because it is odd symmetric)}$$

$$b_n=\frac{2}{\text{®}}\int_{-1/2}^{1/2}2At\sin(\pi\, n\, t)\,dt+$$

$$\frac{2}{\text{®}}\int_{1/2}^{3/2}(2A-2At)\sin(\pi\, n\, t)\,dt$$

$$b_n=\begin{cases}0 & n \text{ is even}\\ \dfrac{8A}{n^2\text{®}^2} & n=1,5,9,13,\ldots\\ -\dfrac{8A}{n^2\text{®}^2} & n=3,7,11,15,\ldots\end{cases}$$

- Fundamental period

  $T_0 = 2$

- Fundamental frequency

  $f_0 = 1/T_0 = 1/2$ Hz

  $w_0 = 2\pi/T_0 = \pi$ rad/s

3/ Performance Objectives :-

Periodic Signals

- ***f*(*t*)** is periodic if, for some positive constant $\boldsymbol{T_0}$

  For all values of t, $f(t) = f(t + T_0)$

  Smallest value of T_0 is the period of $f(t)$.

  $\sin(2\pi f_o t) = \sin(2\pi f_0 t + 2\pi) = \sin(2\pi f_0 t + 4\pi)$: period 2π.

- A periodic signal ***f*(*t*)**

  Unchanged when time-shifted by one period

  Two-sided: extent is $t \in (-\infty, \infty)$

  May be generated by periodically extending one period

  Area under $f(t)$ over any interval of duration equal to the period is the same; e.g., integrating from 0 to T_0 would give the same value as integrating from $-T_0/2$ to $T_0/2$

Sinusoids

- $f_0(t) = C_0 \cos(2\,\pi\, f_0\, t + \theta_0)$
- $f_n(t) = C_n \cos(2\,\pi\, n\, f_0\, t + \theta_n)$
- The frequency, $n\, f_0$, is the nth harmonic of f_0
- Fundamental frequency in Hertz is f_0
- Fundamental frequency in rad/s is $\omega = 2\,\pi\, f_0$
- $C_n \cos(n\,\omega_0\, t + \theta_n) =$
  $C_n \cos(\theta_n) \cos(n\,\omega_0\, t) - C_n \sin(\theta_n) \sin(n\,\omega_0\, t) =$
  $a_n \cos(n\,\omega_0\, t) + b_n \sin(n\,\omega_0\, t)$

Fourier series

The Fourier series exists only when the function f(t) satisfies the following three conditions called Dirichlet conditions

(i) The function f(t) is single valued within the interval T0.

(ii) The function f(t) has a finite number of discontinuities and a finite number of maxima and minima in the interval T0.

(iii) The function f(t) is absolutely integrable, that is,

$$\int_{-T_0}^{T_0} |f(t)|\,dt < \sim$$

Fourier Series

- General representation of a periodic signal

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\mu_0 t) + b_n \sin(n\mu_0 t)$$

- Fourier series coefficients

$$a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)\,dt$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\mu_0 t)\,dt$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(n\mu_0 t)\,dt$$

- Compact Fourier series

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\mu_0 t + \bar{\ }_n)$$

where $c_0 = a_0,\ c_n = \sqrt{a_n^2 + b_n^2}$, and

$$\bar{\ }_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

Existence of the Fourier Series

- Existence $\int_0^{T_0} |f(t)|\, dt < \infty$
- Convergence for all *t* $|f(t)| < \infty \quad \forall t$
- Finite number of maxima and minima in one period of ***f*(*t*)**

Periodic Signals

- For all *t*, ***x*(*t* + *T*) = *x*(*t*)**

  x(*t*) is a period signal

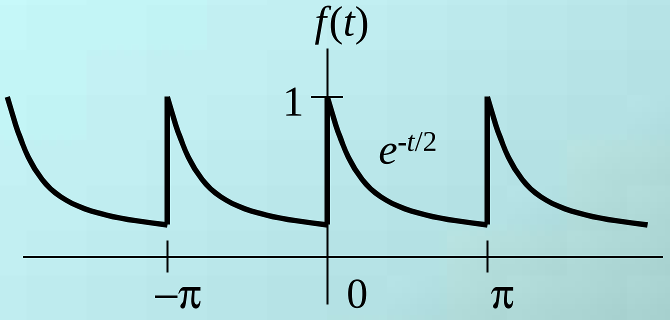
- Periodic signals have a Fourier series representation

$$x(t) = \sum_{m=-\infty}^{\infty} C_n \, e^{j2\pi\left(\frac{m}{T}\right)t}$$

$$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-j2\pi\left(\frac{m}{T}\right)t}\, dt$$

- ***C_n*** computes the projection (components) of ***x*(*t*)** having a frequency that is a multiple of the fundamental frequency **1/*T***.

Example #1



- Fundamental period

  $T_0 = \pi$

- Fundamental frequency

  $f_0 = 1/T_0 = 1/\pi$ Hz

  $w_0 = 2\pi/T_0 = 2$ rad/s

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nt) + b_n \sin(2nt)$$

$$a_0 = \frac{1}{\text{®}}\int_0^{\text{®}} e^{-\frac{t}{2}} dt = -\frac{2}{\text{®}}\left(e^{-\frac{\text{®}}{2}} - 1\right) \approx 0.504$$

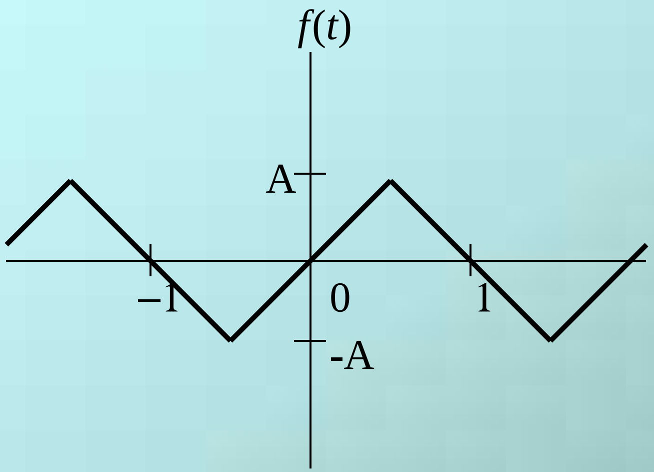
$$a_n = \frac{2}{\text{®}}\int_0^{\text{®}} e^{-\frac{t}{2}} \cos(2nt)\, dt = 0.504\left(\frac{2}{1+16n^2}\right)$$

$$b_n = \frac{2}{\text{®}}\int_0^{\text{®}} e^{-\frac{t}{2}} \sin(2nt)\, dt = 0.504\left(\frac{8n}{1+16n^2}\right)$$

a_n and b_n decrease in amplitude as $n \to \infty$.

$$f(t) = 0.504\left[1 + \sum_{n=1}^{\infty} \frac{2}{1+16n^2}\left(\cos(2nt) + 4n\sin(2nt)\right)\right]$$

Example #2



$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\pi\, n\, t) + b_n \sin(\pi\, n\, t)$$

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- Fundamental period

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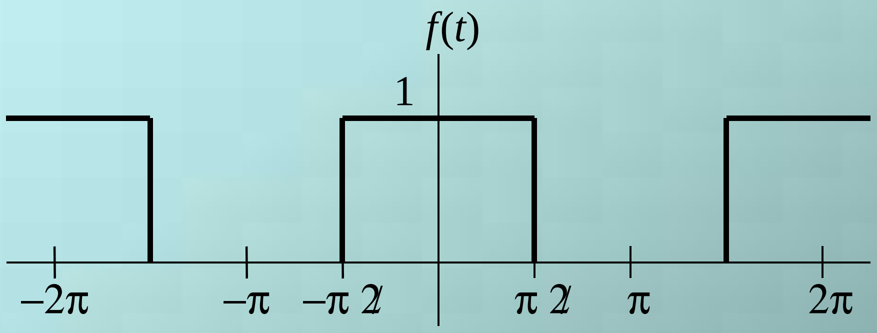
$$b_n = \begin{cases} 0 & n \text{ is even} \\ \dfrac{8A}{n^2 \text{®}^2} & n = 1,5,9,13,\ldots \\ -\dfrac{8A}{n^2 \text{®}^2} & n = 3,7,11,15,\ldots \end{cases}$$

Example #3



- Fundamental period

  $T_0 = 2\pi$

- Fundamental frequency

  $f_0 = 1/T_0 = 1/2\pi$ Hz

  $w_0 = 2\pi/T_0 = 1$ rad/s

$$C_0 = \frac{1}{2}$$

$$C_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{\text{®}\, n} & n \text{ odd} \end{cases}$$

$$\text{–}_n = \begin{cases} 0 & \text{for all } n \neq 3,7,11,15,\ldots \\ -\text{®} & n = 3,7,11,15,\ldots \end{cases}$$

Quiz /

1. What is General representation of a periodic signal

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\mu_0 t) + b_n \sin(n\mu_0 t)$$

2. What are Fourier series coefficients

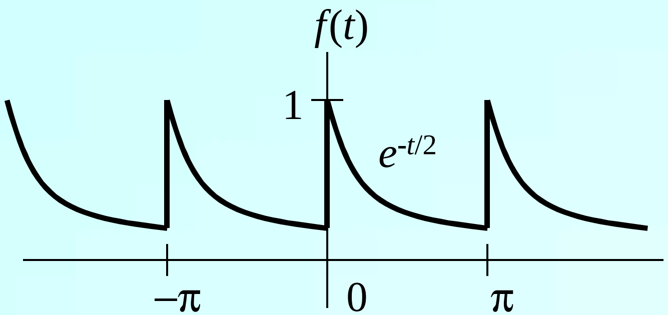
$$a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)dt$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos(n\mu_0 t)dt$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin(n\mu_0 t)dt$$

5/ Post test :-

Find the Fourier series of



Solution

- Fundamental period

  $T_0 = \pi$

- Fundamental frequency

  $f_0 = 1/T_0 = 1/\pi$ Hz

  $w_0 = 2\pi/T_0 = 2$ rad/s

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nt) + b_n \sin(2nt)$$

$$a_0 = \frac{1}{\circledR}\int_0^{\circledR} e^{-\frac{t}{2}} dt = -\frac{2}{\circledR}\left(e^{-\frac{\circledR}{2}} - 1\right) \approx 0.504$$

$$a_n = \frac{2}{\circledR}\int_0^{\circledR} e^{-\frac{t}{2}} \cos(2nt)\, dt = 0.504\left(\frac{2}{1+16n^2}\right)$$

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a_n and b_n decrease in amplitude as $n \to \infty$.

$$f(t) = 0.504\left[1 + \sum_{n=1}^{\infty} \frac{2}{1+16n^2}\left(\cos(2nt) + 4n\sin(2nt)\right)\right]$$

References

1. T. R. Ganesh Babu, and G. Srinivasan: “ Communication Theory and systems”, 2006.

2. Sanjay Sharma: “Communication Systems (Analog and Digital) ”

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