HEAT TRANSFER

انتقال الحرارة

اعداد الدكتور

علي شاكر باقر الجابري

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CHAPTER FIVE NUMERICAL METHODS I N HEAT CONDUCTION

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So far we have mostly considered relatively simple heat conduction problems involving *simple geometries* with simple boundary conditions because only such simple problems can be solved *analytically.* But many problems encountered in practice involve *complicated geometries* with complex boundary conditions or variable properties and cannot be solved analytically. In such cases, sufficiently accurate approximate solutions can be obtained by computers using a *numerical method.*

Analytical solution methods such as those presented in Chapter 2 are based on solving the governing differential equation together with the boundary conditions. They result in solution functions for the temperature at *every point* in the medium. Numerical methods, on the other hand, are based on replacing the differential equation by a set of *n* algebraic equations for the unknown temperatures at *n* selected points in the medium, and the simultaneous solution of these equations results in the temperature values at those *discrete points.*

There are several ways of obtaining the numerical formulation of a heat conduction problem, such as the *finite difference* method, the *finite element* method, the *boundary element* method, and the *energy balance* (or control volume) method. Each method has its own advantages and disadvantages, and each is used in practice. In this chapter we will use primarily the *energy balance* approach since it is based on the familiar energy balances on control volumes instead of heavy mathematical formulations, and thus it gives a better physical feel for the problem. Besides, it results in the same set of algebraic equations as the finite difference method. In this chapter, the numerical formulation and solution of heat conduction problems are demonstrated for both steady and transient cases in various geometries.

5–1 FINITE DIFFERENCE FORMULATION OF DIFFERENTIAL EQUATIONS

The numerical methods for solving differential equations are based on replacing the *differential equations* by *algebraic equations.* In the case of the popular **finite difference** method, this is done by replacing the *derivatives* by *differences.*

Derivatives are the building blocks of differential equations, and thus we first give a brief review of derivatives. Consider a function f that depends on x, as shown in Figure 5.1 . The **first derivative** of $f(x)$ at a point is equivalent to the *slope* of a line tangent to the curve at that point and is defined as

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad 5.1$$

which is the ratio of the increment Δf of the function to the increment Δx of the independent variable as $\Delta x \to 0$. If we don't take the indicated limit, we will have the following *approximate* relation for the derivative:

$$\frac{df(x)}{dx} \cong \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad 5.2$$

Now consider steady one-dimensional heat transfer in a plane wall of thickness L with heat generation. The wall is subdivided into M sections of equal thickness $\Delta x = L/M$ in the x-direction, separated by planes passing through $M + 1$ points $0, 1, 2, \ldots, m - 1, m, m + 1, \ldots, M$ called **nodes** or **nodal points,** as shown in Figure 5–7. The x-coordinate of any point m is simply $x_m = m\Delta x$, and the temperature at that point is simply $T(x_m) = T_m$.

The heat conduction equation involves the second derivatives of temperature with respect to the space variables, such as d^2T/dx^2, and the finite difference formulation is based on replacing the second derivatives by appropriate

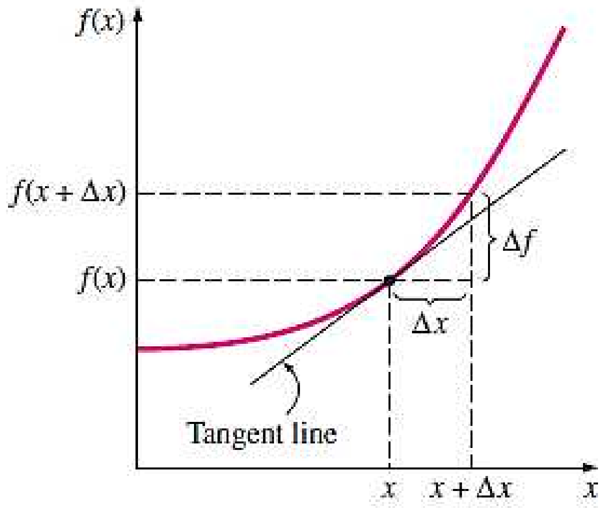
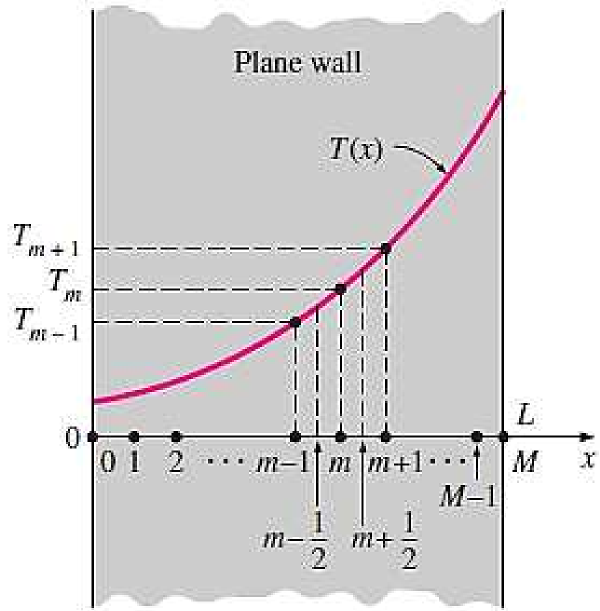


FIGURE 5–1



differences. But we need to start the process with first derivatives. Using Eq. 5.1, the first derivative of temperature dT/dx at the midpoints $m - \frac{1}{2}$ and $m + \frac{1}{2}$ of the sections surrounding the node m can be expressed as

$$\left.\frac{dT}{dx}\right|_{m-\frac{1}{2}} \cong \frac{T_m - T_{m-1}}{\Delta x} \qquad \text{and} \qquad \left.\frac{dT}{dx}\right|_{m+\frac{1}{2}} \cong \frac{T_{m+1} - T_m}{\Delta x}$$

Noting that the second derivative is simply the derivative of the first derivative, the second derivative of temperature at node m can be expressed as

$$\left.\frac{d^2T}{dx^2}\right|_m \cong \frac{\left.\frac{dT}{dx}\right|_{m+\frac{1}{2}} - \left.\frac{dT}{dx}\right|_{m-\frac{1}{2}}}{\Delta x} = \frac{\frac{T_{m+1} - T_m}{\Delta x} - \frac{T_m - T_{m-1}}{\Delta x}}{\Delta x}$$

$$= \frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2}$$

which is the *finite difference representation* of the *second derivative* at a general internal node m. Note that the second derivative of temperature at a node m is expressed in terms of the temperatures at node m and its two neighboring nodes. Then the differential equation

$$\frac{d^2T}{dx^2} + \frac{\dot{g}}{k} = 0$$

which is the governing equation for *steady one-dimensional* heat transfer in a plane wall with heat generation and constant thermal conductivity, can be expressed in the *finite difference* form as (Fig. 5–8)

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0, \qquad m = 1, 2, 3, \ldots, M - 1$$

where $\dot{g}_m$ is the rate of heat generation per unit volume at node m.

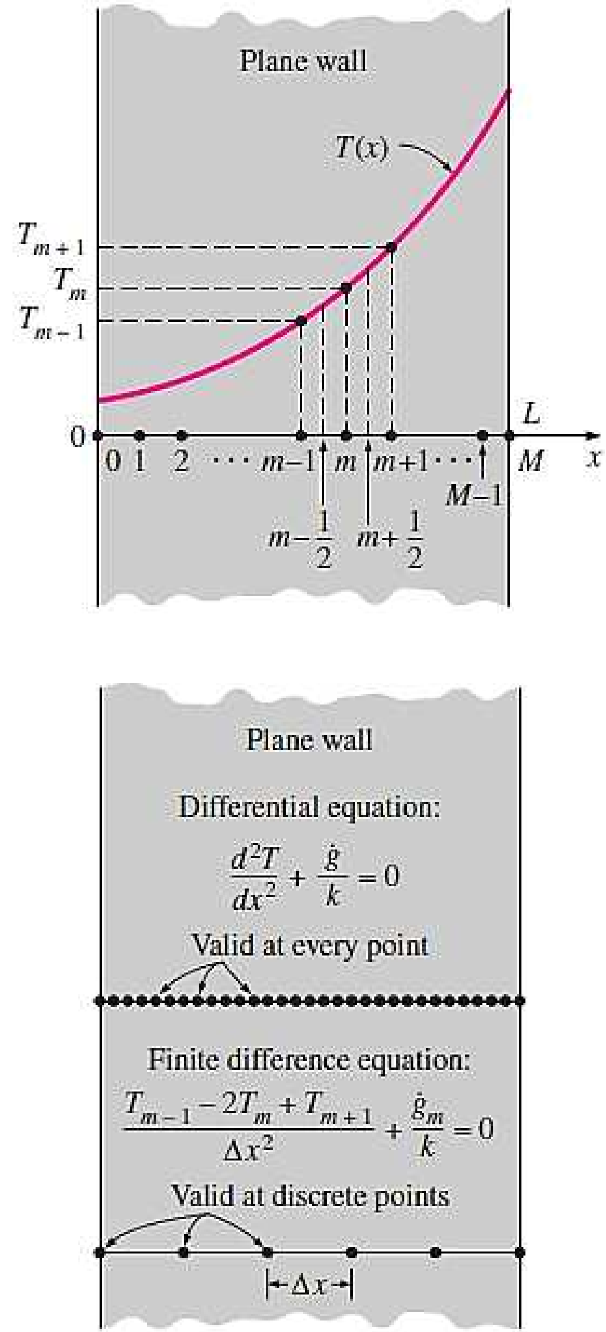


FIGURE 5–8

5–2 ONE-DIMENSIONAL STEADY HEAT CONDUCTION

To demonstrate the approach, again consider steady one-dimensional heat transfer in a plane wall of thickness L with heat generation $\dot{g}(x)$ and constant conductivity k. The wall is now subdivided into M equal regions of thickness $\Delta x = L/M$ in the x-direction, and the divisions between the regions are selected as the nodes. Therefore, we have $M + 1$ nodes labeled 0, 1, 2, . . . , $m - 1$, m, $m + 1$, . . . , M, as shown in Figure 5–10. The x-coordinate of any node m is simply $x_m = m\Delta x$, and the temperature at that point is $T(x_m) = T_m$. Elements are formed by drawing vertical lines through the midpoints between the nodes. Note that all interior elements represented by interior nodes are full-size elements (they have a thickness of Δx), whereas the two elements at the boundaries are half-sized.

To obtain a general difference equation for the interior nodes, consider the element represented by node m and the two neighboring nodes $m - 1$ and $m + 1$. Assuming the heat conduction to be *into* the element on all surfaces, an *energy balance* on the element can be expressed as

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at the left}\\ \text{surface}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at the right}\\ \text{surface}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{pmatrix} \quad \text{or}$$

$$\dot{Q}_{\text{cond, left}} + \dot{Q}_{\text{cond, right}} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

since the energy content of a medium (or any part of it) does not change under *steady* conditions and thus $\Delta E_{\text{element}} = 0$. The rate of *heat generation* within the element can be expressed as

$$\dot{G}_{\text{element}} = \dot{g}_m V_{\text{element}} = \dot{g}_m A\Delta x$$

where $\dot{g}_m$ is the rate of heat generation per unit volume in W/m³ evaluated at node m and treated as a constant for the entire element, and A is heat transfer area, which is simply the inner (or outer) surface area of the wall.

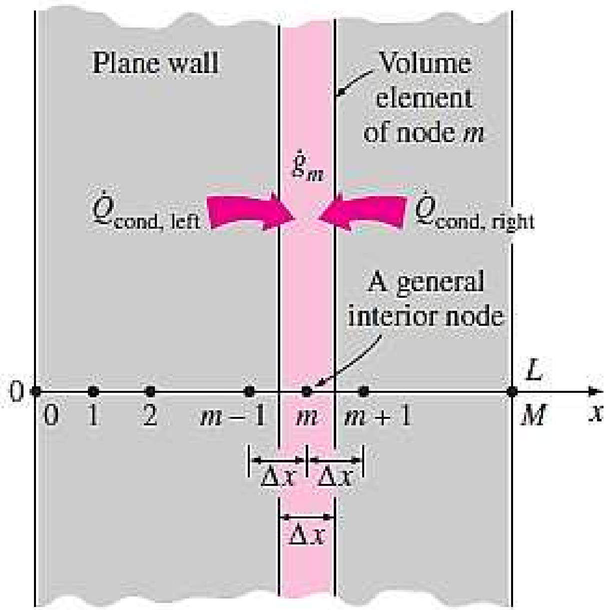
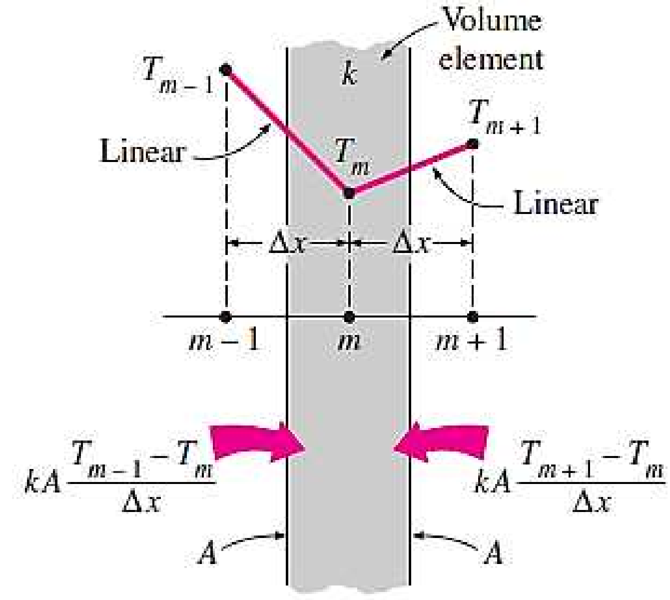


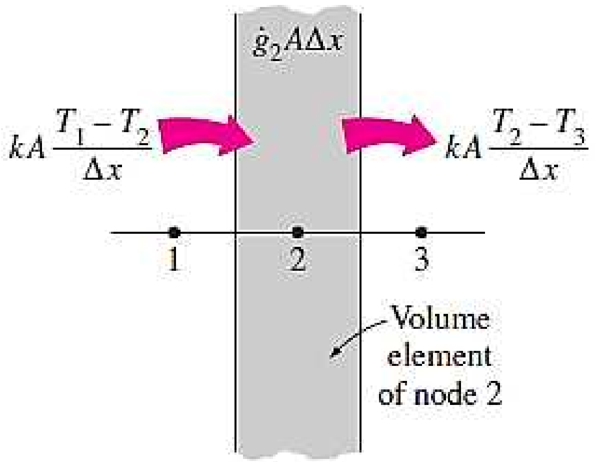
FIGURE 5–10

$$kA\frac{T_{m-1}-T_m}{\Delta x} + kA\frac{T_{m+1}-T_m}{\Delta x} + \dot{g}_m A\Delta x = 0$$

which simplifies to

$$\frac{T_{m-1}-2T_m+T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0, \qquad m = 1, 2, 3, \ldots, M-1$$

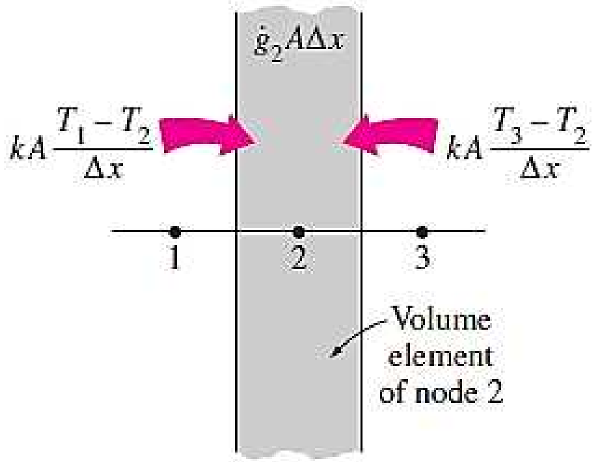




$$kA\frac{T_1-T_2}{\Delta x} - kA\frac{T_2-T_3}{\Delta x} + \dot{g}_2 A\Delta x = 0$$

or

$$T_1 - 2T_2 + T_3 + \dot{g}_2 A\Delta x^2 / k = 0$$



$$kA\frac{T_1-T_2}{\Delta x} + kA\frac{T_3-T_2}{\Delta x} + \dot{g}_2 A\Delta x = 0$$

or

$$T_1 - 2T_2 + T_3 + \dot{g}_2 A\Delta x^2 / k = 0$$

Boundary Conditions

1. **Specified Heat Flux Boundary Condition**

$$\dot{q}_0 A + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$

Special case: **Insulated Boundary** ($\dot{q}_0 = 0$)

$$kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$

2. **Convection Boundary Condition**

$$hA(T_\infty - T_0) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$

3. **Radiation Boundary Condition**

$$\varepsilon\sigma A(T_{surr}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$

4. **Combined Convection and Radiation Boundary Condition**

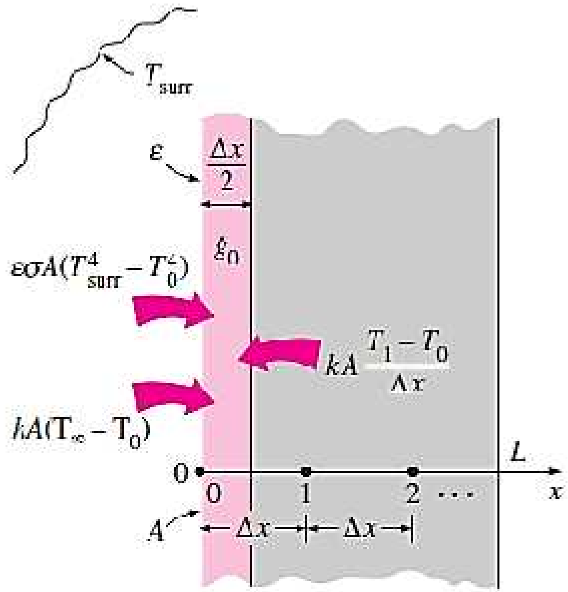
$$hA(T_\infty - T_0) + \varepsilon\sigma A(T_{surr}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$

5. **Combined Convection, Radiation, and Heat Flux Boundary Condition**

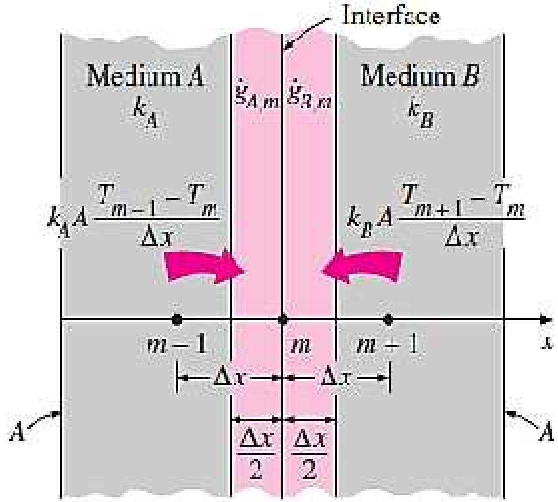
$$\dot{q}_0 A + hA(T_\infty - T_0) + \varepsilon\sigma A(T_{surr}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$

6. **Interface Boundary Condition**

$$k_A A\frac{T_{m-1} - T_m}{\Delta x} + k_B A\frac{T_{m+1} - T_m}{\Delta x} + \dot{g}_{A,m}(A\Delta x/2) + \dot{g}_{B,m}(A\Delta x/2) = 0$$



$$hA(T_\infty - T_0) + \varepsilon\sigma A(T_{surr}^4 - T_0^4)$$
$$+ kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0 A\frac{\Delta x}{2} = 0$$



$$k_A A\frac{T_{m-1} - T_m}{\Delta x} + k_B A\frac{T_{m+1} - T_m}{\Delta x}$$
$$+ \dot{g}_{A,m} A\frac{\Delta x}{2} + \dot{g}_{B,m} A\frac{\Delta x}{2} = 0$$

EXAMPLE 5–1 **Steady Heat Conduction in a Large Uranium Plate**

Consider a large uranium plate of thickness $L = 4$ cm and thermal conductivity $k = 28$ W/m · °C in which heat is generated uniformly at a constant rate of $\dot{g} = 5 \times 10^6$ W/m³. One side of the plate is maintained at 0°C by iced water while the other side is subjected to convection to an environment at $T_\infty = 30$°C with a heat transfer coefficient of $h = 45$ W/m² · °C, as shown in Figure 5–18. Considering a total of three equally spaced nodes in the medium, two at the boundaries and one at the middle, estimate the exposed surface temperature of the plate under steady conditions using the finite difference approach.

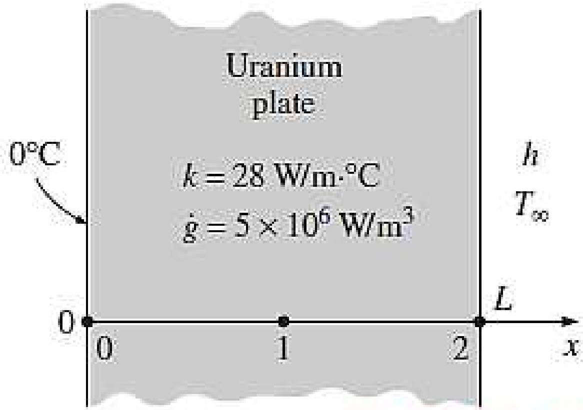


FIGURE 5–18

SOLUTION

Analysis The number of nodes is specified to be $M = 3$, and they are chosen to be at the two surfaces of the plate and the midpoint, as shown in the figure. Then the nodal spacing Δx becomes

$$\Delta x = \frac{L}{M-1} = \frac{0.04 \text{ m}}{3-1} = 0.02 \text{ m}$$

Node 1 is an interior node $m = 1$: $\dfrac{T_0 - 2T_1 + T_2}{\Delta x^2} + \dfrac{\dot{g}_1}{k} = 0 \;\rightarrow\; \dfrac{0 - 2T_1 + T_2}{\Delta x^2} + \dfrac{\dot{g}_1}{k} = 0 \;\rightarrow\; 2T_1 - T_2 = \dfrac{\dot{g}_1 \Delta x^2}{k}$ (1)

Node 2 is a boundary node subjected to convection, $hA(T_\infty - T_2) + kA\dfrac{T_1 - T_2}{\Delta x} + \dot{g}_2(A\Delta x/2) = 0$

Canceling the heat transfer area A and rearranging give $T_1 - \left(1 + \dfrac{h\Delta x}{k}\right)T_2 = -\dfrac{h\Delta x}{k}T_\infty - \dfrac{\dot{g}_2 \Delta x^2}{2k}$ (2)

Equations (1) and (2) form a system of two equations in two unknowns T_1 and T_2. Substituting the given quantities and simplifying gives

$$2T_1 - T_2 = 71.43 \quad \text{(in °C)}$$
$$T_1 - 1.032T_2 = -36.68 \quad \text{(in °C)}$$

This is a system of two algebraic equations in two unknowns and can be solved easily by the elimination method. Solving the first equation for T_1 and substituting into the second equation result in an equation in T_2 whose solution is

$T_2 = $ **136.1°C**

$T_1 = 103.8$°C,

5–3 TWO-DIMENSIONAL STEADY HEAT CONDUCTION

Now consider a *volume element* of size $\Delta x \times \Delta y \times 1$ centered about a general interior node (m, n) in a region in which heat is generated at a rate of $\dot{g}$ and the thermal conductivity k is constant, as shown in Figure 5–24. Again assuming the direction of heat conduction to be *toward* the node under consideration at all surfaces, the energy balance on the volume element can be expressed as

$$\left(\begin{array}{c}\text{Rate of heat conduction}\\ \text{at the left, top, right,}\\ \text{and bottom surfaces}\end{array}\right) + \left(\begin{array}{c}\text{Rate of heat}\\ \text{generation inside}\\ \text{the element}\end{array}\right) = \left(\begin{array}{c}\text{Rate of change of}\\ \text{the energy content}\\ \text{of the element}\end{array}\right) \quad \text{or}$$

$$\dot{Q}_{\text{cond, left}} + \dot{Q}_{\text{cond, top}} + \dot{Q}_{\text{cond, right}} + \dot{Q}_{\text{cond, bottom}} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

for the *steady* case. Again assuming the temperatures between the adjacent nodes to vary linearly and noting that the heat transfer area is $A_x = \Delta y \times 1 = \Delta y$ in the x-direction and $A_y = \Delta x \times 1 = \Delta x$ in the y-direction, the energy balance relation above becomes

$$k\Delta y \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\Delta x \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k\Delta y \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k\Delta x \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + \dot{g}_{m,n}\,\Delta x\,\Delta y = 0$$

Dividing each term by $\Delta x \times \Delta y$ and simplifying gives

$$\frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta y^2} + \frac{\dot{g}_{m,n}}{k} = 0$$

Then $\Delta x = \Delta y = l$, and the relation above simplifies to

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{\dot{g}_{m,n} l^2}{k} = 0 \qquad \text{or} \qquad T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{g}_{\text{node}} l^2}{k} = 0$$

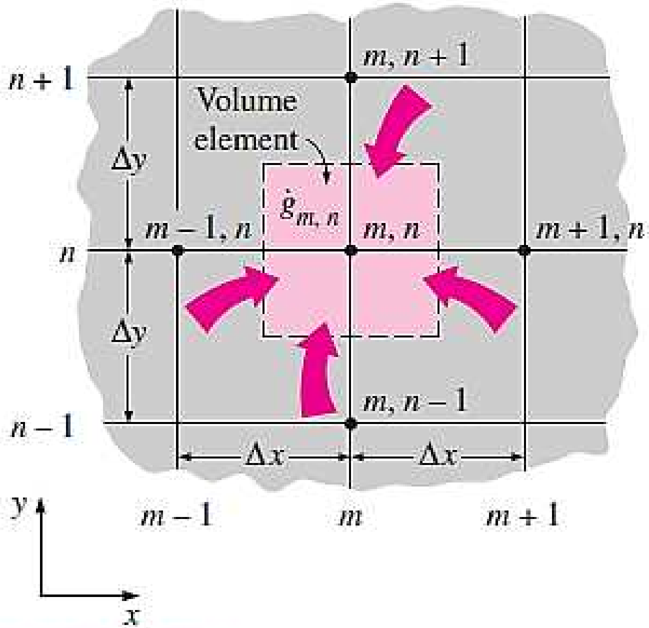


FIGURE 5–24

EXAMPLE 5–3 Steady Two-Dimensional Heat Conduction in L-Bars

Consider steady heat transfer in an L-shaped solid body whose cross section is given in Figure 5–26. Heat transfer in the direction normal to the plane of the paper is negligible, and thus heat transfer in the body is two-dimensional. The thermal conductivity of the body is $k = 15$ W/m · °C, and heat is generated in the body at a rate of $\dot{g} = 2 \times 10^6$ W/m³. The left surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of 90°C. The entire top surface is subjected to convection to ambient air at $T_\infty = 25$°C with a convection coefficient of $h = 80$ W/m² · °C, and the right surface is subjected to heat flux at a uniform rate of $\dot{q}_R = 5000$ W/m². The nodal network of the problem consists of 15 equally spaced nodes with $\Delta x = \Delta y = 1.2$ cm, as shown in the figure. Five of the nodes are at the bottom surface, and thus their temperatures are known. Obtain the finite difference equations at the remaining nine nodes and determine the nodal temperatures by solving them.

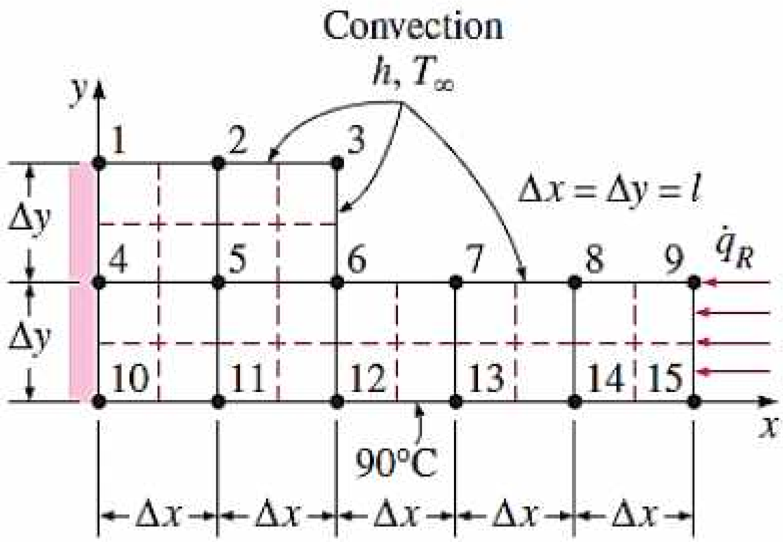


FIGURE 5–26

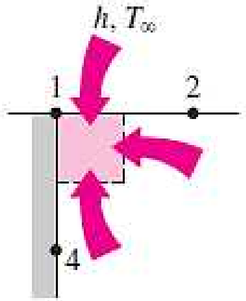
SOLUTION

(a) *Node 1*

$$0 + h\frac{\Delta x}{2}(T_\infty - T_1) + k\frac{\Delta y}{2}\frac{T_2 - T_1}{\Delta x} + k\frac{\Delta x}{2}\frac{T_4 - T_1}{\Delta y} + \dot{g}_1\frac{\Delta x}{2}\frac{\Delta y}{2} = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$-\left(2 + \frac{hl}{k}\right)T_1 + T_2 + T_4 = -\frac{hl}{k}T_\infty - \frac{\dot{g}_1 l^2}{2k}$$



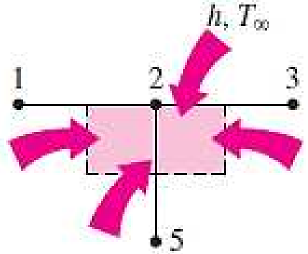
(*a*) Node 1

(b) *Node 2.*

$$h\Delta x(T_\infty - T_2) + k\frac{\Delta y}{2}\frac{T_3 - T_2}{\Delta x} + k\Delta x\frac{T_5 - T_2}{\Delta y} + k\frac{\Delta y}{2}\frac{T_1 - T_2}{\Delta x} + \dot{g}_2\Delta x\frac{\Delta y}{2} = 0$$

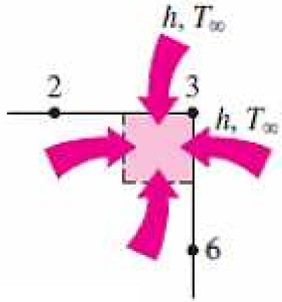
Taking $\Delta x = \Delta y = l$, it simplifies to

$$T_1 - \left(4 + \frac{2hl}{k}\right)T_2 + T_3 + 2T_5 = -\frac{2hl}{k}T_\infty - \frac{\dot{g}_2 l^2}{k}$$



(*b*) Node 2

(*c*) *Node 3.*

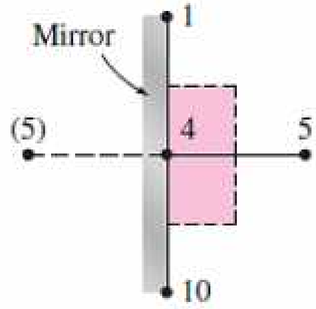


$$h\left(\frac{\Delta x}{2}+\frac{\Delta y}{2}\right)(T_\infty - T_3) + k\frac{\Delta x}{2}\frac{T_6 - T_3}{\Delta y} + k\frac{\Delta y}{2}\frac{T_2 - T_3}{\Delta x} + g_3\frac{\Delta x}{2}\frac{\Delta y}{2} = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$T_2 - \left(2 + \frac{2hl}{k}\right)T_3 + T_6 = -\frac{2hl}{k}T_\infty - \frac{\dot{g}_3 l^2}{2k}$$

(*d*) *Node 4.*

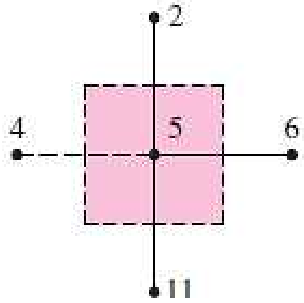


$$T_5 + T_1 + T_5 + T_{10} - 4T_4 + \frac{\dot{g}_4 l^2}{k} = 0$$

or, noting that $T_{10} = 90°$ C,

$$T_1 - 4T_4 + 2T_5 = -90 - \frac{\dot{g}_4 l^2}{k}$$

(*e*) Node 5

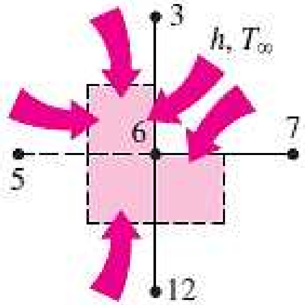


$$T_4 + T_2 + T_6 + T_{11} - 4T_5 + \frac{\dot{g}_5 l^2}{k} = 0$$

or, noting that $T_{11} = 90°C$,

$$T_2 + T_4 - 4T_5 + T_6 = -90 - \frac{\dot{g}_5 l^2}{k}$$

(*f*) *Node 6.*



$$h\left(\frac{\Delta x}{2}+\frac{\Delta y}{2}\right)(T_\infty - T_6) + k\frac{\Delta y}{2}\frac{T_7 - T_6}{\Delta x} + k\Delta x\frac{T_{12} - T_6}{\Delta y}$$
$$+ k\Delta y\frac{T_5 - T_6}{\Delta x} + k\frac{\Delta x}{2}\frac{T_3 - T_6}{\Delta y} + g_6\frac{3\Delta x\Delta y}{4} = 0$$

Taking $\Delta x = \Delta y = l$ and noting that $T_{12} = 90°C$, it simplifies to

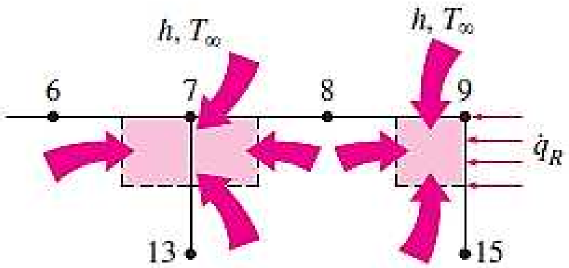
$$T_3 + 2T_5 - \left(6 + \frac{2hl}{k}\right)T_6 + T_7 = -180 - \frac{2hl}{k}T_\infty - \frac{3\dot{g}_6 l^2}{2k}$$

(*g*) *Node 7.*

$$h\Delta x(T_\infty - T_7) + k\frac{\Delta y}{2}\frac{T_8 - T_7}{\Delta x} + k\Delta x\frac{T_{13} - T_7}{\Delta y} + k\frac{\Delta y}{2}\frac{T_6 - T_7}{\Delta x} + \dot{g}_7\Delta x\frac{\Delta y}{2} = 0$$

Taking $\Delta x = \Delta y = l$ and noting that $T_{13} = 90°C$, it simplifies to

$$T_6 - \left(4 + \frac{2hl}{k}\right)T_7 + T_8 = -180 - \frac{2hl}{k}T_\infty - \frac{\dot{g}_7 l^2}{k}$$



(*h*) *Node 8.* This node is identical to Node 7.

$$T_7 - \left(4 + \frac{2hl}{k}\right)T_8 + T_9 = -180 - \frac{2hl}{k}T_\infty - \frac{\dot{g}_8 l^2}{k}$$

(*i*) *Node 9.*

$$h\frac{\Delta x}{2}(T_\infty - T_9) + q_R\frac{\Delta y}{2} + k\frac{\Delta x}{2}\frac{T_{15} - T_9}{\Delta y} + k\frac{\Delta y}{2}\frac{T_8 - T_9}{\Delta x} + g_9\frac{\Delta x}{2}\frac{\Delta y}{2} = 0$$

Taking $\Delta x = \Delta y = l$ and noting that $T_{15} = 90°C$, it simplifies to

$$T_8 - \left(2 + \frac{hl}{k}\right)T_9 = -90 - \frac{\dot{q}_R l}{k} - \frac{hl}{k}T_\infty - \frac{\dot{g}_9 l^2}{2k}$$

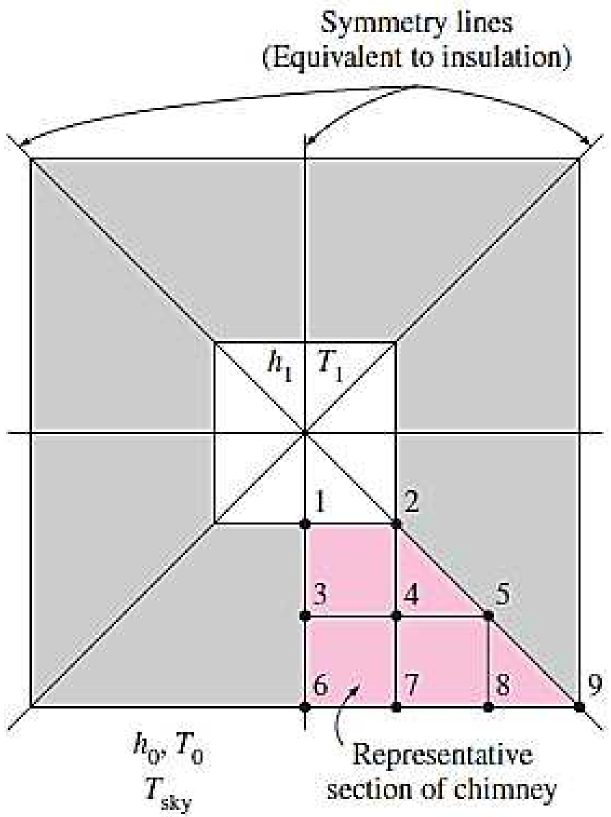
This completes the development of finite difference formulation for this problem. Substituting the given quantities, the system of nine equations for the determination of nine unknown nodal temperatures becomes

$$\begin{aligned}
-2.064T_1 + T_2 + T_4 &= -11.2 \\
T_1 - 4.128T_2 + T_3 + 2T_5 &= -22.4 \\
T_2 - 2.128T_3 + T_6 &= -12.8 \\
T_1 - 4T_4 + 2T_5 &= -109.2 \\
T_2 + T_4 - 4T_5 + T_6 &= -109.2 \\
T_3 + 2T_5 - 6.128T_6 + T_7 &= -212.0 \\
T_6 - 4.128T_7 + T_8 &= -202.4 \\
T_7 - 4.128T_8 + T_9 &= -202.4 \\
T_8 - 2.064T_9 &= -105.2
\end{aligned}$$

$T_1 = 112.1°C$	$T_2 = 110.8°C$	$T_3 = 106.6°C$
$T_4 = 109.4°C$	$T_5 = 108.1°C$	$T_6 = 103.2°C$
$T_7 = 97.3°C$	$T_8 = 96.3°C$	$T_9 = 97.6°C$

EXAMPLE 5–4 Heat Loss through Chimneys

Hot combustion gases of a furnace are flowing through a square chimney made of concrete ($k = 1.4$ W/m · °C). The flow section of the chimney is 20 cm × 20 cm, and the thickness of the wall is 20 cm. The average temperature of the hot gases in the chimney is $T_i = 300$°C, and the average convection heat transfer coefficient inside the chimney is $h_i = 70$ W/m^2 · °C. The chimney is losing heat from its outer surface to the ambient air at $T_o = 20$°C by convection with a heat transfer coefficient of $h_o = 21$ W/m^2 · °C and to the sky by radiation. The emissivity of the outer surface of the wall is $\varepsilon = 0.9$, and the effective sky temperature is estimated to be 260 K. Using the finite difference method with $\Delta x = \Delta y = 10$ cm and taking full advantage of symmetry, determine the temperatures at the nodal points of a cross section and the rate of heat loss for a 1-m-long section of the chimney.



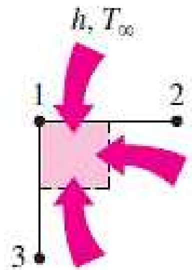
SOLUTION

(*a*) Node 1

$$0 + h_i \frac{\Delta x}{2}(T_i - T_1) + k\frac{\Delta y}{2}\frac{T_2 - T_1}{\Delta x} + k\frac{\Delta x}{2}\frac{T_3 - T_1}{\Delta y} + 0 = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$-\left(2 + \frac{h_i l}{k}\right)T_1 + T_2 + T_3 = -\frac{h_i l}{k}T_i$$

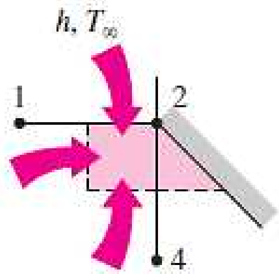


(*b*) Node 2

$$k\frac{\Delta y}{2}\frac{T_1 - T_2}{\Delta x} + h_i\frac{\Delta x}{2}(T_i - T_2) + 0 + k\Delta x\frac{T_4 - T_2}{\Delta y} = 0$$

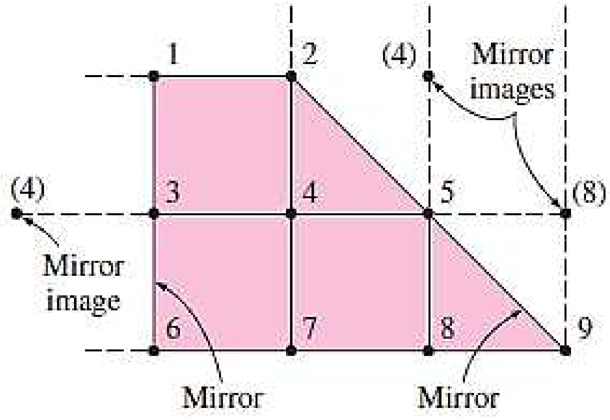
Taking $\Delta x = \Delta y = l$, it simplifies to

$$T_1 - \left(3 + \frac{h_i l}{k}\right)T_2 + 2T_4 = -\frac{h_i l}{k}T_i$$



(*c*) Nodes 3, 4, *and* 5. (Interior nodes, Fig. 5–34)

$$\text{Node 3: } T_4 + T_1 + T_4 + T_6 - 4T_3 = 0$$
$$\text{Node 4: } T_3 + T_2 + T_5 + T_7 - 4T_4 = 0$$
$$\text{Node 5: } T_4 + T_4 + T_8 + T_8 - 4T_5 = 0$$



(*d*) Node 6. (On the outer boundary, subjected to convection and radiation)

$$0 + k\frac{\Delta x}{2}\frac{T_3 - T_6}{\Delta y} + k\frac{\Delta y}{2}\frac{T_7 - T_6}{\Delta x} + h_o\frac{\Delta x}{2}(T_o - T_6) + \varepsilon\sigma\frac{\Delta x}{2}(T_{\text{sky}}^4 - T_6^4) = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$T_2 + T_3 - \left(2 + \frac{h_o l}{k}\right)T_6 = -\frac{h_o l}{k}T_o - \frac{\varepsilon\sigma l}{k}(T_{\text{sky}}^4 - T_6^4)$$

(*d*) Node 6. (On the outer boundary, subjected to convection and radiation)

$$0 + k\frac{\Delta x}{2}\frac{T_3 - T_6}{\Delta y} + k\frac{\Delta y}{2}\frac{T_7 - T_6}{\Delta x} + h_o\frac{\Delta x}{2}(T_o - T_6) + \varepsilon\sigma\frac{\Delta x}{2}(T_{\text{sky}}^4 - T_6^4) = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

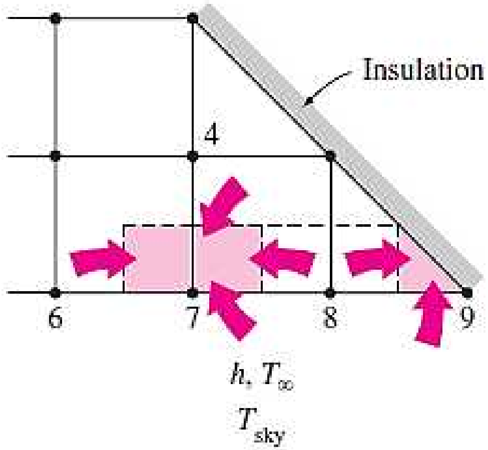
$$T_2 + T_3 - \left(2 + \frac{h_o l}{k}\right)T_6 = -\frac{h_o l}{k}T_o - \frac{\varepsilon\sigma l}{k}(T_{\text{sky}}^4 - T_6^4)$$

(*e*) Node 7. (On the outer boundary, subjected to convection and radiation,

$$k\frac{\Delta y}{2}\frac{T_6 - T_7}{\Delta x} + k\Delta x\frac{T_4 - T_7}{\Delta y} + k\frac{\Delta y}{2}\frac{T_8 - T_7}{\Delta x} + h_o\Delta x(T_o - T_7) + \varepsilon\sigma\Delta x(T_{\text{sky}}^4 - T_7^4) = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$2T_4 + T_6 - \left(4 + \frac{2h_o l}{k}\right)T_7 + T_8 = -\frac{2h_o l}{k}T_o - \frac{2\varepsilon\sigma l}{k}(T_{\text{sky}}^4 - T_7^4)$$



(*f*) Node 8. Same as Node 7, except shift the node numbers up by 1 (replace 4 by 5, 6 by 7, 7 by 8, and 8 by 9 in the last relation)

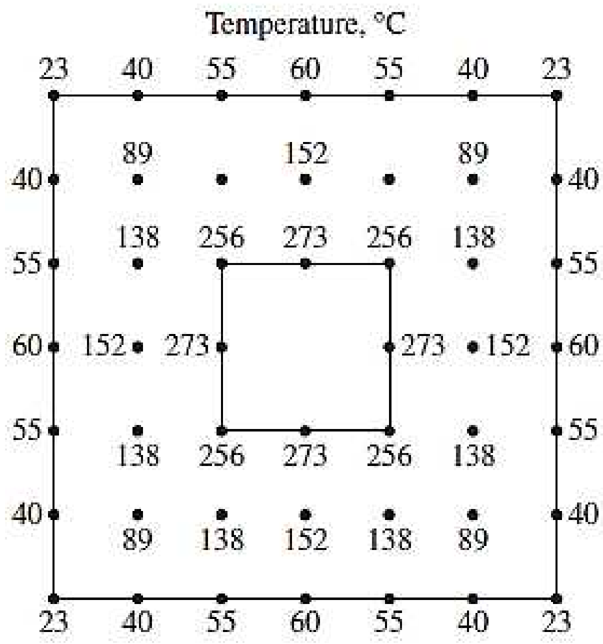
$$2T_5 + T_7 - \left(4 + \frac{2h_o l}{k}\right)T_8 + T_9 = -\frac{2h_o l}{k}T_o - \frac{2\varepsilon\sigma l}{k}(T_{\text{sky}}^4 - T_8^4)$$

(*g*) Node 9. (On the outer boundary, subjected to convection and radiation,

$$k\frac{\Delta y}{2}\frac{T_8 - T_9}{\Delta x} + 0 + h_o\frac{\Delta x}{2}(T_o - T_9) + \varepsilon\sigma\frac{\Delta x}{2}(T_{\text{sky}}^4 - T_9^4) = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$T_8 - \left(1 + \frac{h_o l}{k}\right)T_9 = -\frac{h_o l}{k}T_o - \frac{\varepsilon\sigma l}{k}(T_{\text{sky}}^4 - T_9^4)$$



$$T_1 = (T_2 + T_3 + 2865)/7$$
$$T_2 = (T_1 + 2T_4 + 2865)/8$$
$$T_3 = (T_1 + 2T_4 + T_6)/4$$
$$T_4 = (T_2 + T_3 + T_5 + T_7)/4$$
$$T_5 = (2T_4 + 2T_8)/4$$
$$T_6 = (T_2 + T_3 + 456.2 - 0.3645 \times 10^{-9}\, T_6^4)/3.5$$
$$T_7 = (2T_4 + T_6 + T_8 + 912.4 - 0.729 \times 10^{-9}\, T_7^4)/7$$
$$T_8 = (2T_5 + T_7 + T_9 + 912.4 - 0.729 \times 10^{-9}\, T_8^4)/7$$
$$T_9 = (T_8 + 456.2 - 0.3645 \times 10^{-9}\, T_9^4)/2.5$$

$T_1 = 545.7$ K $= 272.6$°C	$T_2 = 529.2$ K $= 256.1$°C	$T_3 = 425.2$ K $= 152.1$°C
$T_4 = 411.2$ K $= 138.0$°C	$T_5 = 362.1$ K $= 89.0$°C	$T_6 = 332.9$ K $= 59.7$°C
$T_7 = 328.1$ K $= 54.9$°C	$T_8 = 313.1$ K $= 39.9$°C	$T_9 = 296.5$ K $= 23.4$°C

5–4 TRANSIENT HEAT CONDUCTION

Transient Heat Conduction in a Plane Wall

$$\sum_{\text{All sides}} \dot{Q} + \dot{G}_{\text{element}} = \rho V_{\text{element}} C \frac{T_m^{i+1} - T_m^i}{\Delta t} \qquad (5\text{-}39)$$

where T_m^i and T_m^{i+1} are the temperatures of node m at times $t_i = i\Delta t$ and $t_{i+1} = (i+1)\Delta t$, respectively, and $T_m^{i+1} - T_m^i$ represents the temperature change of the node during the time interval Δt between the time steps i and $i + 1$

Consider transient one-dimensional heat conduction in a plane wall of thickness L with heat generation $\dot{g}(x, t)$ that may vary with time and position and constant conductivity k with a mesh size of $\Delta x = L/M$ and nodes 0, 1, 2, . . . , M in the x-direction, as shown in Figure 5–40. Noting that the volume element of a general interior node m involves heat conduction from two sides and the volume of the element is $V_{\text{element}} = A\Delta x$, the transient finite difference formulation for an interior node can be expressed on the basis of Eq. 5–39 as

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + \dot{g}_m A\Delta x = \rho A\Delta x C\frac{T_m^{i+1} - T_m^i}{\Delta t}$$

Canceling the surface area A and multiplying by $\Delta x/k$, it simplifies to

$$T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{g}_m \Delta x^2}{k} = \frac{\Delta x^2}{\alpha \Delta t}(T_m^{i+1} - T_m^i)$$

where $\alpha = k/\rho C$ is the *thermal diffusivity* of the wall material. We now define a dimensionless **mesh Fourier number** as

$$\tau = \frac{\alpha \Delta t}{\Delta x^2}$$

$$T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{g}_m \Delta x^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

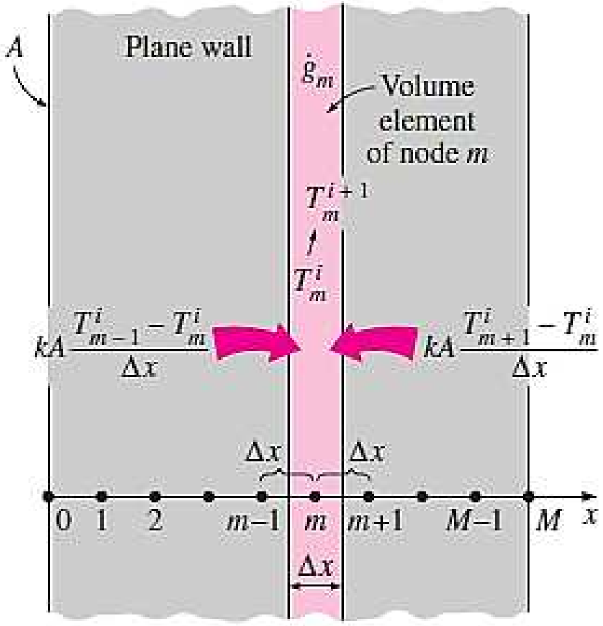


FIGURE 5–40

the stability criterion for all nodes in this case is $1 - 2\tau \geq 0$ or $\tau = \frac{\alpha \Delta t}{\Delta x^2} \leq \frac{1}{2}$

EXAMPLE 5–5 Transient Heat Conduction in a Large Uranium Plate

Consider a large uranium plate of thickness $L = 4$ cm, thermal conductivity $k = 28$ W/m · °C, and thermal diffusivity $\alpha = 12.5 \times 10^{-6}$ m²/s that is initially at a uniform temperature of 200°C. Heat is generated uniformly in the plate at a constant rate of $\dot{g} = 5 \times 10^6$ W/m³. At time $t = 0$, one side of the plate is brought into contact with iced water and is maintained at 0°C at all times, while the other side is subjected to convection to an environment at $T_\infty = 30$°C with a heat transfer coefficient of $h = 45$ W/m² · °C, as shown in Figure 5–44. Considering a total of three equally spaced nodes in the medium, two at the boundaries and one at the middle, estimate the exposed surface temperature of the plate 2.5 min after the start of cooling using (*a*) the explicit method and (*b*) the implicit method.

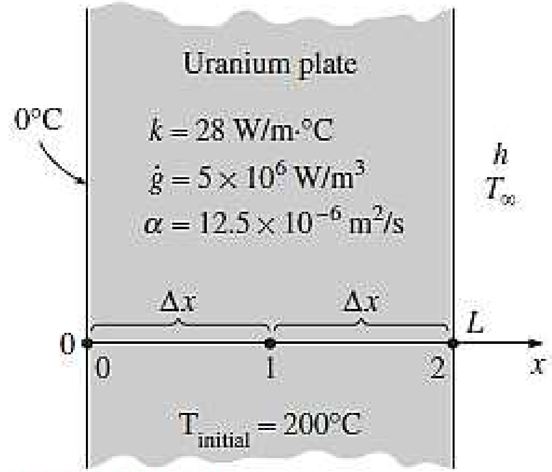


FIGURE 5–44

SOLUTION

$$\Delta x = \frac{L}{M-1} = \frac{0.04 \text{ m}}{3-1} = 0.02 \text{ m}$$

(*a*) Node 1 is an interior node

$$T_1^{i+1} = \tau(T_0 + T_2^i) + (1 - 2\tau)\,T_1^i + \tau\frac{\dot{g}_1 \Delta x^2}{k} \qquad (1)$$

Node 2 is a boundary node subjected to convection

$$hA(T_\infty - T_2^i) + kA\frac{T_1^i - T_2^i}{\Delta x} + \dot{g}_2 A\frac{\Delta x}{2} = \rho A\frac{\Delta x}{2} C\frac{T_2^{i+1} - T_2^i}{\Delta x}$$

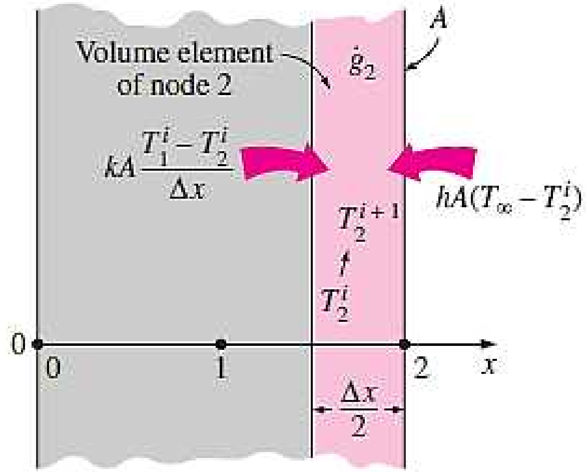
Dividing by $kA/2\Delta x$ and using the definitions of thermal diffusivity $\alpha = k/\rho C$ and the dimensionless mesh Fourier number $\tau = \alpha\Delta t/(\Delta x)^2$ gives

$$\frac{2h\Delta x}{k}(T_\infty - T_2^i) + 2(T_1^i - T_2^i) + \frac{\dot{g}_2 \Delta x^2}{k} = \frac{T_2^{i+1} - T_2^i}{\tau}$$

which can be solved for T_2^{i+1} to give

$$T_2^{i+1} = \left(1 - 2\tau - 2\tau\frac{h\Delta x}{k}\right)T_2^i + \tau\left(2T_1^i + 2\frac{h\Delta x}{k}T_\infty + \frac{\dot{g}_2 \Delta x^2}{k}\right) \qquad (2)$$

$$1 - 2\tau - 2\tau\frac{h\Delta x}{k} \geq 0 \quad \rightarrow \quad \tau \leq \frac{1}{2(1 + h\Delta x/k)} \quad \rightarrow \quad \Delta t \leq \frac{\Delta x^2}{2\alpha(1 + h\Delta x/k)}$$



since $\tau = \alpha\Delta t/(\Delta x)^2$. Substituting the given quantities, the maximum allowable value of the time step is determined to be

$$\Delta t \le \frac{(0.02\ \text{m})^2}{2(12.5 \times 10^{-6}\ \text{m}^2/\text{s})[1 + (45\ \text{W/m}^2 \cdot {}^\circ\text{C})(0.02\ \text{m})/28\ \text{W/m} \cdot {}^\circ\text{C}]} = 15.5\ \text{s}$$

Therefore, any time step less than 15.5 s can be used to solve this problem. For convenience, let us choose the time step to be $\Delta t = 15$ s. Then the mesh Fourier number becomes

$$\tau = \frac{\alpha\Delta t}{(\Delta x)^2} = \frac{(12.5 \times 10^{-6}\ \text{m}^2/\text{s})(15\ \text{s})}{(0.02\ \text{m})^2} = 0.46875 \quad (\text{for } \Delta t = 15\ \text{s})$$

Substituting this value of τ and other given quantities, the explicit finite difference equations (1) and (2) developed here reduce to

$$T_1^{i+1} = 0.0625T_1^i + 0.46875T_2^i + 33.482$$
$$T_2^{i+1} = 0.9375T_1^i + 0.032366T_2^i + 34.386$$

The initial temperature of the medium at $t = 0$ and $i = 0$ is given to be 200°C throughout, and thus $T_1^0 = T_2^0 = 200°\text{C}$. Then the nodal temperatures at T_1^1 and T_2^1 at $t = \Delta t = 15$ s are determined from these equations to be

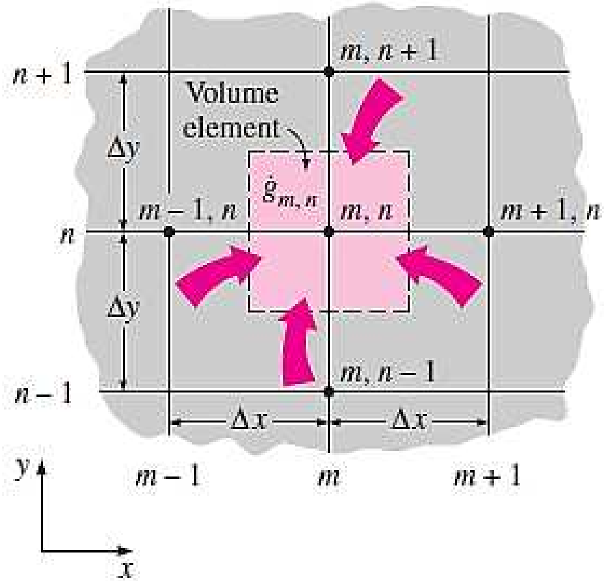
$$\begin{aligned} T_1^1 &= 0.0625T_1^0 + 0.46875T_2^0 + 33.482 \\ &= 0.0625 \times 200 + 0.46875 \times 200 + 33.482 = 139.7°\text{C} \\ T_2^1 &= 0.9375T_1^0 + 0.032366T_2^0 + 34.386 \\ &= 0.9375 \times 200 + 0.032366 \times 200 + 34.386 = 228.4°\text{C} \end{aligned}$$

Time Step, i	Time, s	Node Temperature, °C T_1^i	T_2^i
0	0	200.0	200.0
1	15	139.7	228.4
2	30	149.3	172.8
3	45	123.8	179.9
4	60	125.6	156.3
5	75	114.6	157.1
6	90	114.3	146.9
7	105	109.5	146.3
8	120	108.9	141.8
9	135	106.7	141.1
10	150	106.3	139.0
20	300	103.8	136.1
30	450	103.7	136.0
40	600	103.7	136.0

Two-Dimensional Transient Heat Conduction

$$k\Delta y \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\Delta x \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k\Delta y \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$+ k\Delta x \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + \dot{g}_{m,n}\,\Delta x \Delta y = \rho \Delta x \Delta y\, C \frac{T_m^{i+1} - T_m^i}{\Delta t}$$



Taking a square mesh ($\Delta x = \Delta y = l$) and dividing each term by k gives after simplifying,

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{\dot{g}_{m,n} l^2}{k} = \frac{T_m^{i+1} - T_m^i}{\tau}$$

where again $\alpha = k/\rho C$ is the thermal diffusivity of the material and $\tau = \alpha \Delta t / l^2$ is the dimensionless mesh Fourier number. It can also be expressed in terms of the temperatures at the neighboring nodes in the following easy-to-remember form:

$$T_{\text{left}}^i + T_{\text{top}}^i + T_{\text{right}}^i + T_{\text{bottom}}^i - 4T_{\text{node}}^i + \frac{\dot{g}_{\text{node}}^i l^2}{k} = \frac{T_{\text{node}}^{i+1} - T_{\text{node}}^i}{\tau}$$

stability criterion for all interior nodes in this case is $1 - 4\tau > 0$, or

$$\tau = \frac{\alpha \Delta t}{l^2} \le \frac{1}{4}$$

EXAMPLE 5–7 Transient Two-Dimensional Heat Conduction in L-Bars

Consider two-dimensional transient heat transfer in an L-shaped solid body that is initially at a uniform temperature of 90°C and whose cross section is given in Figure 5–51. The thermal conductivity and diffusivity of the body are k = 15 W/m · °C and $\alpha = 3.2 \times 10^{-6}$ m²/s, respectively, and heat is generated in the body at a rate of $\dot{g} = 2 \times 10^6$ W/m³. The left surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of 90°C at all times. At time $t = 0$, the entire top surface is subjected to convection to ambient air at $T_\infty = 25$°C with a convection coefficient of h = 80 W/m² · °C, and the right surface is subjected to heat flux at a uniform rate of $\dot{q}_R = 5000$ W/m². The nodal network of the problem consists of 15 equally spaced nodes with $\Delta x = \Delta y = 1.2$ cm, as shown in the figure. Five of the nodes are at the bottom surface, and thus their temperatures are known. Using the explicit method, determine the temperature at the top corner (node 3) of the body after 1, 3, 5, 10, and 60 min.

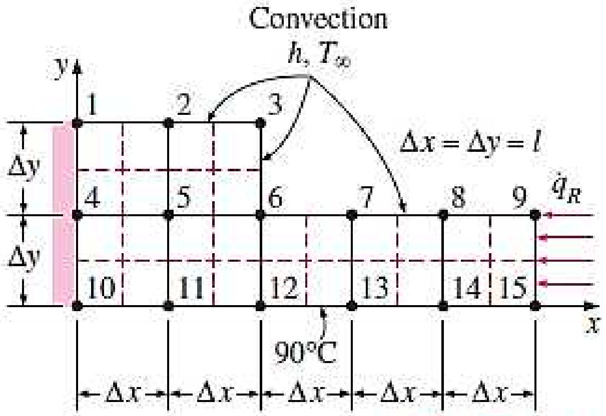


FIGURE 5–51

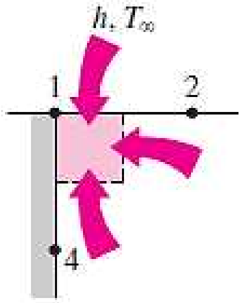
SOLUTION

$$\sum_{\text{All sides}} \dot{Q}^i + \dot{G}^i_{\text{element}} = \rho V_{\text{element}} C \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

Fourier number $\tau = \alpha \Delta t / l^2$, where $\Delta x = \Delta y = l$.

(*a*) Node 1. (Boundary node subjected to convection and insulation,

$$h\frac{\Delta x}{2}(T_\infty - T_1^i) + k\frac{\Delta y}{2}\frac{T_2^i - T_1^i}{\Delta x} + k\frac{\Delta x}{2}\frac{T_4^i - T_1^i}{\Delta y} + \dot{g}_1 \frac{\Delta x}{2}\frac{\Delta y}{2} = \rho \frac{\Delta x}{2}\frac{\Delta y}{2} C \frac{T_1^{i+1} - T_1^i}{\Delta t}$$



(*a*) Node 1

Dividing by $k/4$ and simplifying,

$$\frac{2hl}{k}(T_\infty - T_1^i) + 2(T_2^i - T_1^i) + 2(T_4^i - T_1^i) + \frac{\dot{g}_1 l^2}{k} = \frac{T_1^{i+1} - T_1^i}{\tau}$$

which can be solved for T_1^{i+1} to give

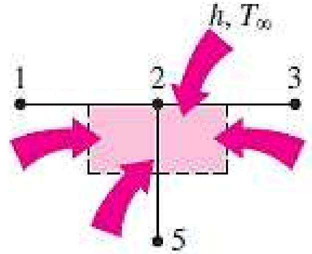
$$T_1^{i+1} = \left(1 - 4\tau - 2\tau\frac{hl}{k}\right)T_1^i + 2\tau\left(T_2^i + T_4^i + \frac{hl}{k}T_\infty + \frac{\dot{g}_1 l^2}{2k}\right)$$

(*b*) Node 2. (Boundary node subjected to convection,

$$h\Delta x(T_\infty - T_2^i) + k\frac{\Delta y}{2}\frac{T_3^i - T_2^i}{\Delta x} + k\Delta x\frac{T_5^i - T_2^i}{\Delta y} + k\frac{\Delta y}{2}\frac{T_1^i - T_2^i}{\Delta x} + \dot{g}_2\,\Delta x\frac{\Delta y}{2} = \rho\Delta x\frac{\Delta y}{2}C\frac{T_2^{i+1} - T_2^i}{\Delta t}$$

Dividing by $k/2$, simplifying, and solving for T_2^{i+1} gives

$$T_2^{i+1} = \left(1 - 4\tau - 2\tau\frac{hl}{k}\right)T_2^i + \tau\left(T_1^i + T_3^i + 2T_5^i + \frac{2hl}{k}T_\infty + \frac{\dot{g}_2 l^2}{k}\right)$$



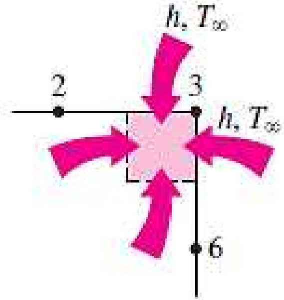
(*b*) Node 2

(*c*) Node 3. (Boundary node subjected to convection on two sides,

$$h\left(\frac{\Delta x}{2} + \frac{\Delta y}{2}\right)(T_\infty - T_3^i) + k\frac{\Delta x}{2}\frac{T_6^i - T_3^i}{\Delta y} + k\frac{\Delta y}{2}\frac{T_2^i - T_3^i}{\Delta x} + \dot{g}_3\frac{\Delta x}{2}\frac{\Delta y}{2} = \rho\frac{\Delta x}{2}\frac{\Delta y}{2}\frac{T_3^{i+1} - T_3^i}{\Delta t}$$

Dividing by $k/4$, simplifying, and solving for T_3^{i+1} gives

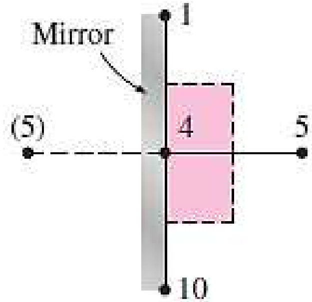
$$T_3^{i+1} = \left(1 - 4\tau - 4\tau\frac{hl}{k}\right)T_3^i + 2\tau\left(T_4^i + T_6^i + 2\frac{hl}{k}T_\infty + \frac{\dot{g}_3 l^2}{2k}\right)$$



(*a*) Node 3

(*d*) Node 4. (On the insulated boundary, and can be treated as an interior node, Noting that $T_{10} = 90°C$

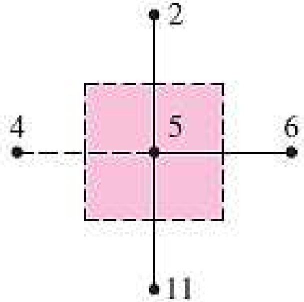
$$T_4^{i+1} = (1 - 4\tau)\,T_4^i + \tau\left(T_1^i + 2T_5^i + 90 + \frac{\dot{g}_4 l^2}{k}\right)$$



(*b*) Node 4

(*e*) Node 5. (Interior node, Noting that $T_{11} = 90°C$,

$$T_5^{i+1} = (1 - 4\tau)\,T_5^i + \tau\left(T_2^i + T_4^i + T_6^i + 90 + \frac{\dot{g}_5 l^2}{k}\right)$$



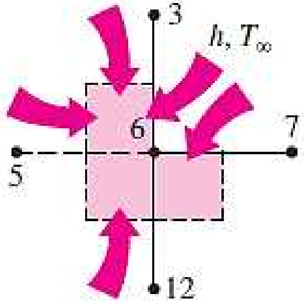
(*a*) Node 5

(*f*) Node 6. (Boundary node subjected to convection on two sides,

$$h\left(\frac{\Delta x}{2}+\frac{\Delta y}{2}\right)(T_\infty - T_6^i) + k\frac{\Delta y}{2}\frac{T_7^i - T_6^i}{\Delta x} + k\Delta x\frac{T_{12}^i - T_6^i}{\Delta y} + k\Delta y\frac{T_5^i - T_6^i}{\Delta x}$$
$$+\frac{\Delta x}{2}\frac{T_3^i - T_6^i}{\Delta y} + g_6\frac{3\Delta x\Delta y}{4} = \rho\frac{3\Delta x\Delta y}{4}C\frac{T_6^{i+1} - T_6^i}{\Delta t}$$

Dividing by $3k/4$, simplifying, and solving for T_6^{i+1} gives

$$T_6^{i+1} = \left(1 - 4\tau - 4\tau\frac{hl}{3k}\right)T_3^i$$
$$+\frac{\tau}{3}\left[2T_3^i + 4T_5^i + 2T_7^i + 4\times 90 + 4\frac{hl}{k}T_\infty + 3\frac{\dot{g}_6 l^2}{k}\right]$$



(*b*) Node 6

(*g*) Node 7. (Boundary node subjected to convection

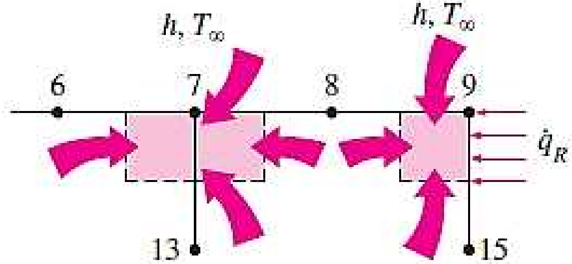
$$h\Delta x(T_\infty - T_7^i) + k\frac{\Delta y}{2}\frac{T_8^i - T_7^i}{\Delta x} + k\Delta x\frac{T_{13}^i - T_7^i}{\Delta y}$$
$$+k\frac{\Delta y}{2}\frac{T_6^i - T_7^i}{\Delta x} + g_7\Delta x\frac{\Delta y}{2} = \rho\Delta x\frac{\Delta y}{2}C\frac{T_7^{i+1} - T_7^i}{\Delta t}$$

Dividing by $k/2$, simplifying, and solving for T_7^{i+1} gives

$$T_7^{i+1} = \left(1 - 4\tau - 2\tau\frac{hl}{k}\right)T_7^i + \tau\left[T_6^i + T_8^i + 2\times 90 + \frac{2hl}{k}T_\infty + \frac{\dot{g}_7 l^2}{k}\right]$$

(*h*) Node 8. This node is identical to node 7

$$T_8^{i+1} = \left(1 - 4\tau - 2\tau\frac{hl}{k}\right)T_8^i + \tau\left[T_7^i + T_9^i + 2\times 90 + \frac{2hl}{k}T_\infty + \frac{\dot{g}_8 l^2}{k}\right]$$



(*i*) Node 9. (Boundary node subjected to convection on two sides,

$$h\frac{\Delta x}{2}(T_\infty - T_9^i) + q_R\frac{\Delta y}{2}k\frac{\Delta x}{2}\frac{T_{15}^i - T_9^i}{\Delta y}$$
$$+\frac{k\Delta y}{2}\frac{T_8^i - T_9^i}{\Delta x} + \dot{g}_9\frac{\Delta x}{2}\frac{\Delta y}{2} = \rho\frac{\Delta x}{2}\frac{\Delta y}{2}C\frac{T_9^{i+1} - T_9^i}{\Delta t}$$

Dividing by $k/4$, simplifying, and solving for T_9^{i+1} gives

$$T_9^{i+1} = \left(1 - 4\tau - 2\tau\frac{hl}{k}\right)T_9^i + 2\tau\left(T_8^i + 90 + \frac{\dot{q}_R l}{k} + \frac{hl}{k}T_\infty + \frac{\dot{g}_9 l^2}{2k}\right)$$

$$1 - 4\tau - 4\tau\frac{hl}{k} \geq 0 \quad \rightarrow \quad \tau \leq \frac{1}{4(1 + hl/k)} \quad \rightarrow \quad \Delta t \leq \frac{l^2}{4\alpha(1 + hl/k)}$$

since $\tau = \alpha\Delta t/l^2$. Substituting the given quantities, the maximum allowable value of the time step is determined to be

$$\Delta t \leq \frac{(0.012\text{ m})^2}{4(3.2 \times 10^{-6}\text{ m}^2/\text{s})[1 + (80\text{ W/m}^2\cdot{}^\circ\text{C})(0.012\text{ m})/(15\text{ W/m}\cdot{}^\circ\text{C})]} = 10.6\text{ s}$$

Therefore, any time step less than 10.6 s can be used to solve this problem. For convenience, let us choose the time step to be $\Delta t = 10$ s. Then the mesh Fourier number becomes

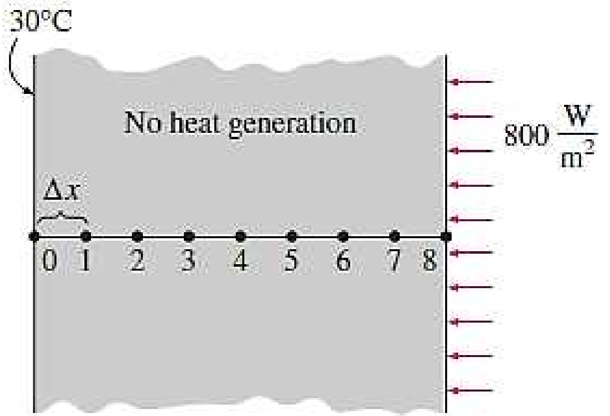
$$\tau = \frac{\alpha\Delta t}{l^2} = \frac{(3.2 \times 10^{-6}\text{ m}^2/\text{s})(10\text{ s})}{(0.012\text{ m})^2} = 0.222 \qquad (\text{for } \Delta t = 10\text{ s})$$

Substituting this value of τ and other given quantities, the developed transient finite difference equations simplify to

$$T_1^{i+1} = 0.0836T_1^i + 0.444(T_2^i + T_4^i + 11.2)$$
$$T_2^{i+1} = 0.0836T_2^i + 0.222(T_1^i + T_3^i + 2T_5^i + 22.4)$$
$$T_3^{i+1} = 0.0552T_3^i + 0.444(T_2^i + T_6^i + 12.8)$$
$$T_4^{i+1} = 0.112T_4^i + 0.222(T_1^i + 2T_5^i + 109.2)$$
$$T_5^{i+1} = 0.112T_1^i + 0.222(T_2^i + T_4^i + T_6^i + 109.2)$$
$$T_6^{i+1} = 0.0931T_6^i + 0.074(2T_3^i + 4T_5^i + 2T_7^i + 424)$$
$$T_7^{i+1} = 0.0836T_7^i + 0.222(T_6^i + T_8^i + 202.4)$$
$$T_8^{i+1} = 0.0836T_8^i + 0.222(T_7^i + T_9^i + 202.4)$$
$$T_9^{i+1} = 0.0836T_9^i + 0.444(T_8^i + 105.2)$$

Using the specified initial condition as the solution at time $t = 0$ (for $i = 0$), sweeping through these nine equations will give the solution at intervals of 10 s. The solution at the upper corner node (node 3) is determined to be 100.2, 105.9, 106.5, 106.6, and 106.6°C at 1, 3, 5, 10, and 60 min, respectively.

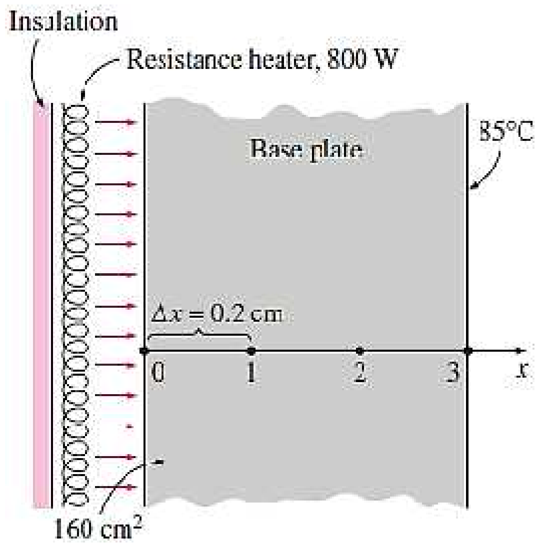
5–16 Consider steady heat conduction in a plane wall whose left surface (node 0) is maintained at 30°C while the right surface (node 8) is subjected to a heat flux of 800 W/m². Express the finite difference formulation of the boundary nodes 0 and 8 for the case of no heat generation. Also obtain the finite difference formulation for the rate of heat transfer at the left boundary.



5–24 Consider a large uranium plate of thickness 5 cm and thermal conductivity $k = 28$ W/m · °C in which heat is generated uniformly at a constant rate of $\dot{g} = 6 \times 10^5$ W/m³. One side of the plate is insulated while the other side is subjected to convection to an environment at 30°C with a heat transfer coefficient of $h = 60$ W/m² · °C. Considering six equally spaced nodes with a nodal spacing of 1 cm, (*a*) obtain the finite difference formulation of this problem and (*b*) determine the nodal temperatures under steady conditions by solving those equations.

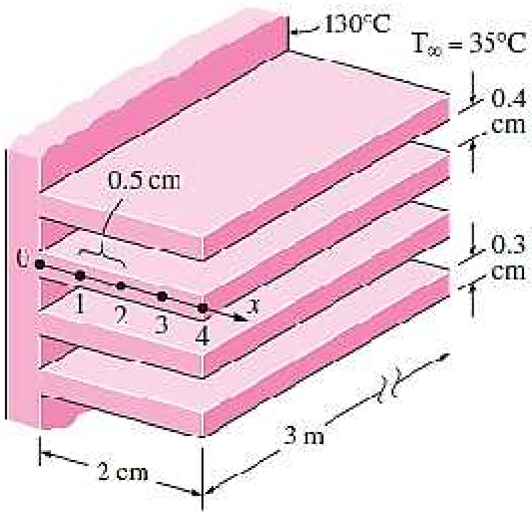
5–27 Consider a large plane wall of thickness $L = 0.4$ m, thermal conductivity $k = 2.3$ W/m · °C, and surface area $A = 20$ m². The left side of the wall is maintained at a constant temperature of 80°C, while the right side loses heat by convection to the surrounding air at $T_\infty = 15$°C with a heat transfer coefficient of $h = 24$ W/m² · °C. Assuming steady one-dimensional heat transfer and taking the nodal spacing to be 10 cm, (*a*) obtain the finite difference formulation for all nodes, (*b*) determine the nodal temperatures by solving those equations, and (*c*) evaluate the rate of heat transfer through the wall.

5–28 Consider the base plate of a 800-W household iron having a thickness of $L = 0.6$ cm, base area of $A = 160$ cm², and thermal conductivity of $k = 20$ W/m · °C. The inner surface of the base plate is subjected to uniform heat flux generated by the resistance heaters inside. When steady operating conditions are reached, the outer surface temperature of the plate is measured to be 85°C. Disregarding any heat loss through the upper part of the iron and taking the nodal spacing to be 0.2 cm, (*a*) obtain the finite difference formulation for the nodes and (*b*) determine the inner surface temperature of the plate by solving those equations. *Answer:* (*b*) 100°C

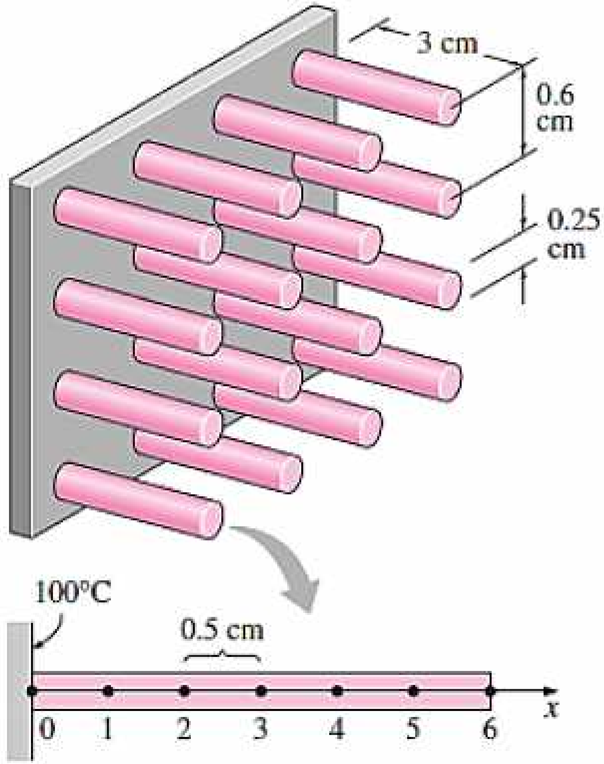


5–29 Consider a large plane wall of thickness $L = 0.3$ m, thermal conductivity $k = 2.5$ W/m · °C, and surface area $A = 12$ m^2. The left side of the wall is subjected to a heat flux of $\dot{q}_0 = 700$ W/m^2 while the temperature at that surface is measured to be $T_0 = 60$°C. Assuming steady one-dimensional heat transfer and taking the nodal spacing to be 6 cm, (*a*) obtain the finite difference formulation for the six nodes and (*b*) determine the temperature of the other surface of the wall by solving those equations.

5–35 One side of a 2-m-high and 3-m-wide vertical plate at 130°C is to be cooled by attaching aluminum fins (k = 237 W/m · °C) of rectangular profile in an environment at 35°C. The fins are 2 cm long, 0.3 cm thick, and 0.4 cm apart. The heat transfer coefficient between the fins and the surrounding air for combined convection and radiation is estimated to be 30 W/m^2 · °C. Assuming steady one-dimensional heat transfer along the fin and taking the nodal spacing to be 0.5 cm, determine (*a*) the finite difference formulation of this problem, (*b*) the nodal temperatures along the fin by solving these equations, (*c*) the rate of heat transfer from a single fin,



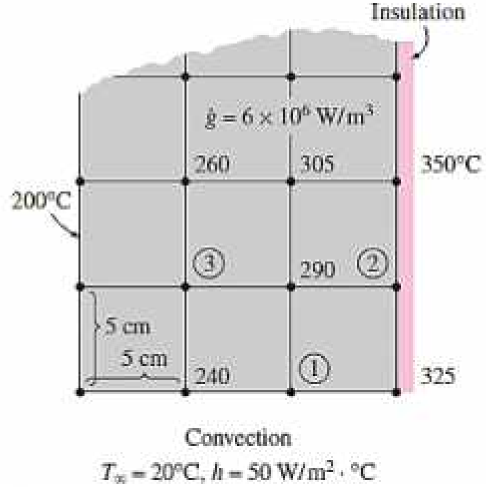
5–36 A hot surface at 100°C is to be cooled by attaching 3-cm-long, 0.25-cm-diameter aluminum pin fins (k = 237 W/m · °C) with a center-to-center distance of 0.6 cm. The temperature of the surrounding medium is 30°C, and the combined heat transfer coefficient on the surfaces is 35 W/m^2 · °C. Assuming steady one-dimensional heat transfer along the fin and taking the nodal spacing to be 0.5 cm, determine (*a*) the finite difference formulation of this problem, (*b*) the nodal temperatures along the fin by solving these equations, (*c*) the rate of heat transfer from a single fin, and (*d*) the rate of heat transfer from a 1-m × 1-m section of the plate.



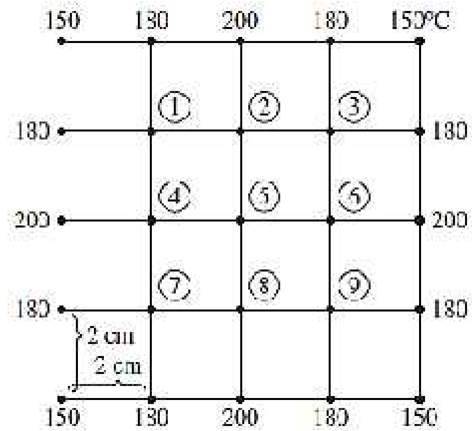
5–37 Repeat Problem 5–36 using copper fins (k = 386 W/m · °C) instead of aluminum ones.

Answers: (*b*) 98.6°C, 97.5°C, 96.7°C, 96.0°C, 95.7°C, 95.5°C

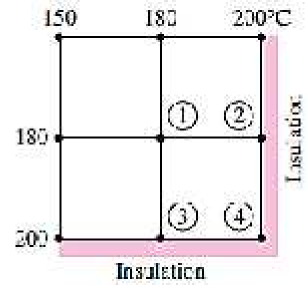
5–46 Consider steady two-dimensional heat transfer in a long solid body whose cross section is given in the figure. The temperatures at the selected nodes and the thermal conditions at the boundaries are as shown. The thermal conductivity of the body is $k = 45$ W/m · °C, and heat is generated in the body uniformly at a rate of $\dot{g} = 6 \times 10^6$ W/m³. Using the finite difference method with a mesh size of $\Delta x = \Delta y = 5.0$ cm, determine (*a*) the temperatures at nodes 1, 2, and 3 and (*b*) the rate of heat loss from the bottom surface through a 1-m-long section of the body.



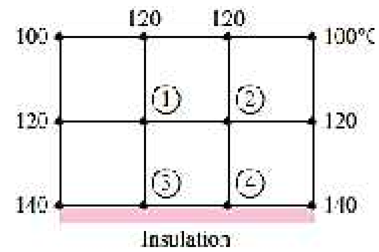
5–47 Consider steady two-dimensional heat transfer in a long solid body whose cross section is given in the figure. The measured temperatures at selected points of the outer surfaces are as shown. The thermal conductivity of the body is $k = 45$ W/m · °C, and there is no heat generation. Using the finite difference method with a mesh size of $\Delta x = \Delta y = 2.0$ cm, determine the temperatures at the indicated points in the medium. *Hint:* Take advantage of symmetry.



5–48 Consider steady two-dimensional heat transfer in a long solid bar whose cross section is given in the figure. The measured temperatures at selected points of the outer surfaces are as shown. The thermal conductivity of the body is $k = 20$ W/m · °C, and there is no heat generation. Using the finite difference method with a mesh size of $\Delta x = \Delta y = 1.0$ cm, determine the temperatures at the indicated points in the medium. *Answers:* $T_1 = 185$°C, $T_2 = T_3 = T_4 = 190$°C

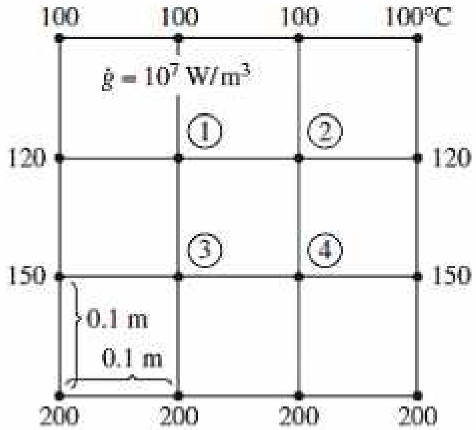


(*a*)

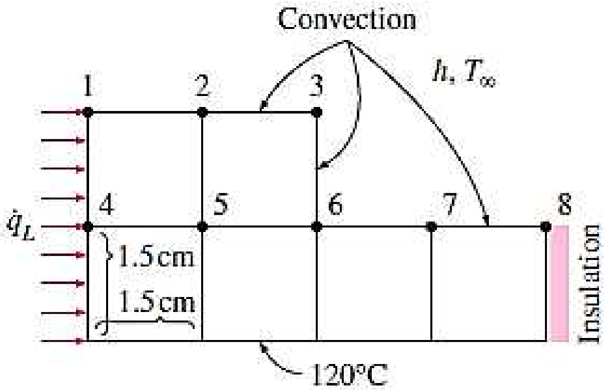


(*b*)

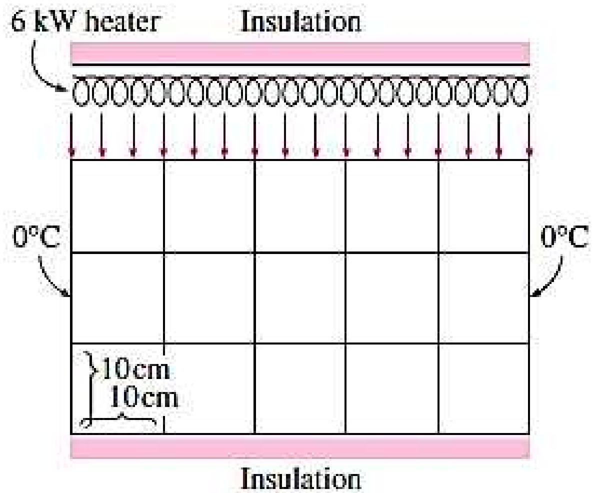
5–50 Consider steady two-dimensional heat transfer in a long solid body whose cross section is given in the figure. The temperatures at the selected nodes and the thermal conditions on the boundaries are as shown. The thermal conductivity of the body is $k = 180$ W/m · °C, and heat is generated in the body uniformly at a rate of $\dot{g} = 10^7$ W/m^3. Using the finite difference method with a mesh size of $\Delta x = \Delta y = 10$ cm, determine (*a*) the temperatures at nodes 1, 2, 3, and 4 and (*b*) the rate of heat loss from the top surface through a 1-m-long section of the body.



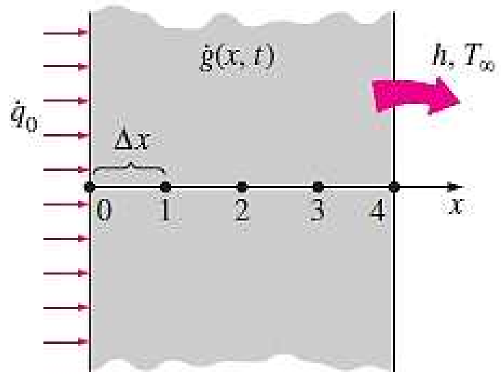
5–53 Consider steady two-dimensional heat transfer in an L-shaped solid body whose cross section is given in the figure. The thermal conductivity of the body is $k = 45$ W/m · °C, and heat is generated in the body at a rate of $\dot{g} = 5 \times 10^6$ W/m^3.



5–62 Consider a 5-m-long constantan block ($k = 23$ W/m · °C) 30 cm high and 50 cm wide. The block is completely submerged in iced water at 0°C that is well stirred, and the heat transfer coefficient is so high that the temperatures on both sides of the block can be taken to be 0°C. The bottom surface of the bar is covered with a low-conductivity material so that heat transfer through the bottom surface is negligible. The top surface of the block is heated uniformly by a 6-kW resistance heater. Using the finite difference method with a mesh size of $\Delta x = \Delta y = 10$ cm and taking advantage of symmetry, (*a*) obtain the finite difference formulation of this problem for steady two-dimensional heat transfer, (*b*) determine the unknown nodal temperatures by solving those equations, and (*c*) determine the rate of heat transfer from the block to the iced water.

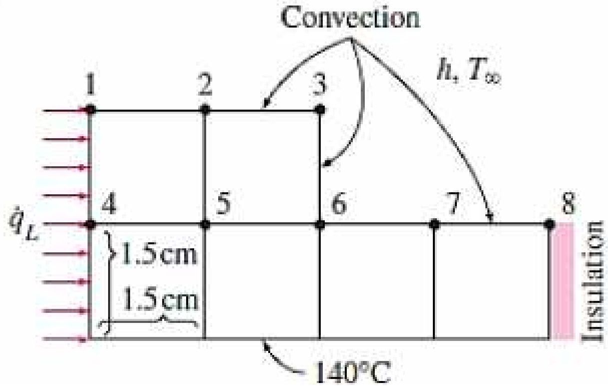


5–74 Consider transient heat conduction in a plane wall with variable heat generation and constant thermal conductivity. The nodal network of the medium consists of nodes 0, 1, 2, 3, and 4 with a uniform nodal spacing of Δx. The wall is initially at a specified temperature. Using the energy balance approach, obtain the explicit finite difference formulation of the boundary nodes for the case of uniform heat flux $\dot{q}_0$ at the left boundary (node 0) and convection at the right boundary (node 4) with a convection coefficient of h and an ambient temperature of T_∞. Do not simplify.



5–84 Consider a large uranium plate of thickness $L = 8$ cm, thermal conductivity $k = 28$ W/m · °C, and thermal diffusivity $\alpha = 12.5 \times 10^{-6}$ m²/s that is initially at a uniform temperature of 100°C. Heat is generated uniformly in the plate at a constant rate of $\dot{g} = 10^6$ W/m³. At time $t = 0$, the left side of the plate is insulated while the other side is subjected to convection with an environment at $T_\infty = 20$°C with a heat transfer coefficient of $h = 35$ W/m² · °C. Using the explicit finite difference approach with a uniform nodal spacing of $\Delta x = 2$ cm, determine (*a*) the temperature distribution in the plate after 5 min and (*b*) how long it will take for steady conditions to be reached in the plate.

5–87 Consider two-dimensional transient heat transfer in an L-shaped solid bar that is initially at a uniform temperature of 140°C and whose cross section is given in the figure. The thermal conductivity and diffusivity of the body are $k = 15$ W/m · °C and $\alpha = 3.2 \times 10^{-6}$ m²/s, respectively, and heat is generated in the body at a rate of $\dot{g} = 2 \times 10^7$ W/m³. The right surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of 140°C at all times. At time $t = 0$, the entire top surface is subjected to convection with ambient air at $T_\infty = 25$°C with a heat transfer coefficient of $h = 80$ W/m² · °C, and the left surface is subjected to uniform heat flux at a rate of $\dot{q}_L = 8000$ W/m². The nodal network of the problem consists of 13 equally spaced nodes with $\Delta x = \Delta y = 1.5$ cm. Using the explicit method, determine the temperature at the top corner (node 3) of the body after 2, 5, and 30 min.



5–89 Consider a long solid bar (k = 28 W/m · °C and α = 12 × 10^{-6} m^2/s) of square cross section that is initially at a uniform temperature of 20°C. The cross section of the bar is 20 cm × 20 cm in size, and heat is generated in it uniformly at a rate of $\dot{g}$ = 8 × 10^5 W/m^3. All four sides of the bar are subjected to convection to the ambient air at T_∞ = 30°C with a heat transfer coefficient of h = 45 W/m^2 · °C. Using the explicit finite difference method with a mesh size of Δx = Δy = 10 cm, determine the centerline temperature of the bar (*a*) after 10 min and (*b*) after steady conditions are established.