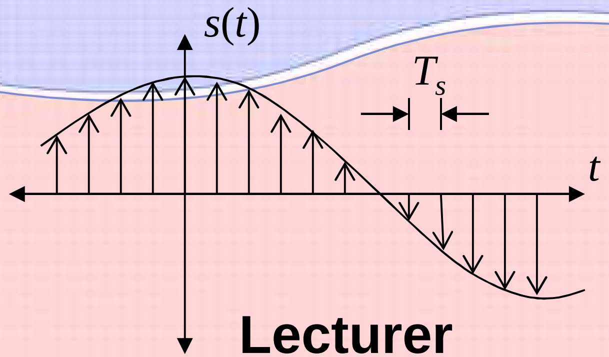

بسم الله الرحمن الرحيم

Ministry of Higher Education and Scientific Research
Foundation of Technical Education
Technical College / Al-Najaf

Training package in

Sampling Theorem

For students of second class



Lecturer
Ahmed H. Hadi

1/ Over view

1 / A –Target population :-

For students of second class in Communications Techniques Engineering Department

1 / B –Rationale :-

A continuous- time signal can be processing its samples through a discrete- time system. For this purpose, it is important to maintain the signal sampling rate sufficiently high so that the original signal can be reconstructed from these samples without error (or with an error within a given tolerance). The necessary quantitative framework for this purpose is provided by the sampling theorem

<u>1 / C –Central Idea :-</u>

a) Sampling theorem

b) Nyquist rate and Nyquist interval

c) Aliasing

<u>1 / D –Objectives:-</u>

Convert a continuous-time signal into discrete-time signal.

2/ Pre test :-

١ What is Aliasing?

Aliasing refers to the phenomenon of a high frequency component in the spectrum of the signal seemingly taking on the identity of a lower frequency in the spectrum of its sampled version.

٢ **What is the reason for occurrence of Aliasing?**

Generally the signal x(t) is strictly band limited, but in practice, an information bearing signal is not strictly band limited, with the result that some degree of undersampling is encountered. Consequently, some aliasing is produced by the sampling process.

3/ Performance Objectives :-

Sampling theorem.

The Sampling Theorem

The sampling theorem may be stated as under:
"A continuous-time signal may be completely represented in its samples and recovered back if the sampling frequency is fs ñ 2 fm. Here fs is the sampling frequency and fm is the maximum frequency present in the signal"

Nyquist rate and Nyquist interval

The sampling rate of 2W samples per seconds, for a signal bandwidth of W Hertz, is called Nyquist rate; its reciprocal (measured in seconds) is called Nyquist interval.

Aliasing

Aliasing refers to the phenomenon of a high frequency component in the spectrum of the signal seemingly taking on the identity of a lower frequency in the spectrum of its sampled version.

The statement of sampling theorem can be given in two parts as:

(i) A band-limited signal of finite energy, which has no frequency-component higher than fm Hz, is completely described by its sample values at uniform intervals less than or equal to $\frac{1}{2f_m}$ **second apart.**

(ii) A band-limited signal of finite energy, which has no frequency components higher than fm Hz, may be completely recovered from the knowledge of its samples taken at the rate of 2 fm samples per second.

Sampling

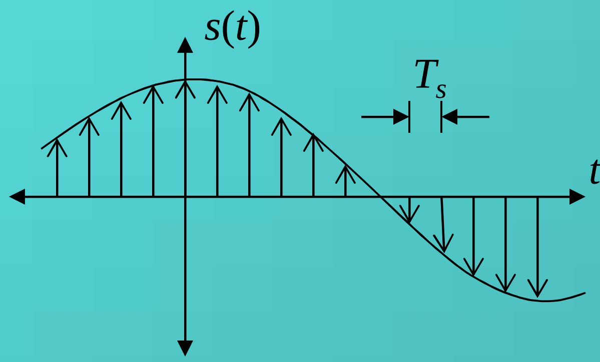
- Many signals originate as continuous-time signals, e.g. conventional music or voice.
- By sampling a continuous-time signal at isolated, equally-spaced points in time, we obtain a sequence of numbers

$$s[k] = s(n\,T_s)$$

– $n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

– *Ts* is the sampling period.

$$s_{sampled}(t) = \sum_{n=-\infty}^{\infty} \underbrace{s(n\,T_s)}_{s[n]}\text{¢}\,(t - n\,T_s)$$



Sampled analog waveform

Shannon Sampling Theorem

- A continuous-time signal $\boldsymbol{x(t)}$ with frequencies no higher than $\boldsymbol{f_{max}}$ can be reconstructed from its samples $\boldsymbol{x[n] = x(n\,T_s)}$ if the samples are taken at a rate $\boldsymbol{f_s}$ which is greater than $\boldsymbol{2\,f_{max}}$.
  - Nyquist rate = $2\,f_{max}$
  - Nyquist frequency = $f_s/2$.
- What happens if $\boldsymbol{f_s = 2f_{max}}$?
- Consider a sinusoid $\boldsymbol{\sin(2\,\pi\,f_{max}\,t)}$
  - Use a sampling period of $T_s = 1/f_s = 1/2f_{max}$.
  - Sketch: sinusoid with zeros at $t = 0, 1/2f_{max}, 1/f_{max}, \ldots$

Sampling Theorem Assumptions

- The continuous-time signal has no frequency content above the frequency f_{max}
- The sampling time is exactly the same between any two samples
- The sequence of numbers obtained by sampling is represented in exact precision
- The conversion of the sequence of numbers to continuous-time is ideal

Why 44.1 kHz for Audio CDs?

- Sound is audible in 20 Hz to 20 kHz range:

  f_{max} = 20 kHz and the Nyquist rate 2 f_{max} = 40 kHz
- What is the extra 10% of the bandwidth used?

  Rolloff from passband to stopband in the magnitude response of the anti-aliasing filter
- Okay, 44 kHz makes sense. Why 44.1 kHz?

  At the time the choice was made, only recorders capable of storing such high rates were VCRs.

  NTSC: 490 lines/frame, 3 samples/line, 30 frames/s = 44100 samples/s

  PAL: 588 lines/frame, 3 samples/line, 25 frames/s = 44100 samples/s

Sampling

- As sampling rate increases, sampled waveform looks more and more like the original.
- Many applications (e.g. communication systems) care more about frequency content in the waveform and not its shape
- Zero crossings: frequency content of a sinusoid
  - Distance between two zero crossings: one half period.
  - With the sampling theorem satisfied, sampled sinusoid crosses zero at the right times even though its waveform shape may be difficult to recognize

Aliasing

- Analog sinusoid

  $x(t) = A\cos(2\pi f_0 t + \phi)$

- Sample at $T_s = 1/f_s$

  $x[n] = x(T_s n) =$
  $A\cos(2\pi f_0 T_s n + \phi)$

- Keeping the sampling period same, sample

  $y(t) = A\cos(2\pi(f_0 + lf_s)t + \phi)$

  where l is an integer

$$
\begin{aligned}
y[n] &= y(T_s n) \\
&= A\cos(2\pi(f_0 + lf_s)T_s n + \phi) \\
&= A\cos(2\pi f_0 T_s n + 2\pi l f_s T_s n + \phi) \\
&= A\cos(2\pi f_0 T_s n + 2\pi l n + \phi) \\
&= A\cos(2\pi f_0 T_s n + \phi) \\
&= x[n]
\end{aligned}
$$

Here, $f_s T_s = 1$

Since l is an integer,
$\cos(x + 2\pi l) = \cos(x)$

- $y[n]$ indistinguishable from $x[n]$

Aliasing

- Since l is any integer, an infinite number of sinusoids will give same sequence of samples.
- The frequencies $f_0 + l f_s$ for $l \div 0$ are called aliases of frequency f_0 with respect f_s to because all of the aliased frequencies appear to be the same as f_0 when sampled by f_s

Generalized Sampling Theorem

- Sampling rate must be greater than twice the bandwidth
  - Bandwidth is defined as the non-zero extent of the spectrum of the continuous-time signal in positive frequencies
  - For a lowpass signal with maximum frequency f_{max}, the bandwidth is f_{max}
  - For a bandpass signal with frequency content on the interval $[f_1, f_2]$, the bandwidth is $f_2 - f_1$

Example:

An analog signal is expressed by the equation

$$x(t) = 3\cos(50\text{®}t) + 10\sin(300\text{®}t) - \cos(100\text{®}t).$$

Calculate the Nyquist rate for this signal.

Solution: The given signal is expressed as

$$x(t) = 3\cos(50\text{®}t) + 10\sin(300\text{®}t) - \cos(100\text{®}t)\ldots\ldots\ldots\ldots(i)$$

Let three frequencies present be $\mu_1, \mu_2 \ \& \ \mu_3$

So that the new equation for signal,

$$x(t) = 3\cos(\mu_1 t) + 10\sin(\mu_2 t) - \cos(\mu_3 t)\ldots\ldots\ldots\ldots(ii)$$

Comparing equations $(i)\ \&\ (ii)$

$$\mu_1 t = 50\,\circledR\, t;\quad \mu_1 = 50\,\circledR$$

$$or\quad 2\circledR f_1 = 50\,\circledR$$

$$or\quad 2\ f_1 = 50$$

$$\therefore\ or\quad f_1 = 25\ Hz$$

Similarly, for second factor

$$\mu_2 t = 300\,\circledR\, t;\quad \mu_2 = 300\,\circledR$$

$$or\ \ 2\circledR f_2 = 300\,\circledR\quad or\ \ 2\circledR f_2 = 300\,\circledR$$

$$\therefore\ f_2 = 150\ Hz\,.$$

Again, for third factor

$$\mu_3 t = 100\circledR t;\ \ or\ 2\circledR f_3 t = 100\circledR t$$

$$2\circledR f_3 = 100\circledR$$

$$\therefore f_3 = 50 Hz.$$

Therefore, the maximum frequency present in x(t) is,

$$f_2 = 150 \ Hz \ .$$

Nyquist rate is given as

$$f_s = 2 f_m$$

where f_m= Maximum frequency present in the signal.

$$f_m = f_2 = 150 Hz$$

Therefore, Nyquist rate

$$f_s = 2 f_2 = 2 \times 150$$

$$f_s = 300 \, Hz$$

Quiz/

What are the various types of sampling.

a. Natural sampling
b. ideal sampling
c. flat-top sampling

5/ Post test :-

1. State sampling theorem

The sampling theorem may be stated as under:

"A continuous-time signal may be completely represented in its samples and recovered back if the sampling frequency is fs ñ 2 fm. Here fs is the sampling frequency and fm is the maximum frequency present in the signal"

2. What is Nyquist rate and Nyquist interval

The sampling rate of 2W samples per seconds, for a signal bandwidth of W Hertz, is called Nyquist rate; its reciprocal (measured in seconds) is called Nyquist interval.

References

1. T. R. Ganesh Babu, and G. Srinivasan: “ Communication Theory and systems”, 2006.

2. Sanjay Sharma: “Communication Systems (Analog and Digital) ”.

شكراً لكم