AL FURAT AL AWSAT TECHNICAL UNIVERSITY

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Linear constant coefficient difference equations

A subclass of linear shift invariant systems are those for which the input x(n) and output y(n) satisfy an N^{th} order linear constant coefficient difference equation, given by

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{r=0}^{M} b_r\, x(n-r)\ , a_0 \neq 0$$

If the system is causal, then we can rearrange the equation above to provide a computational realization of the system as follows:

$$y(n) = -\sum_{k=1}^{N} \frac{a_k}{a_0}\, y(n-k) + \sum_{r=0}^{M} \frac{b_r}{a_0}\, x(n-r)$$

Example

Solve the following difference equation for y(n), assuming y(n) = 0 for all n < 0 and x(n) = δ(n).

$$y(n) - a\, y(n-1) = x(n)$$

This corresponds to calculate the response of the system when excited by an impulse, assuming zero initial conditions.

Solution:

$$y(n) - a\, y(n-1) = x(n)$$

Evaluating when n = 0, $y(0) = a\, y(-1) + x(0) = 1$

n = 1, $y(1) = a\, y(0) + x(1) = a$

n = 2, $y(2) = a\, y(1) + x(2) = a^2$

Continuing this process it is easy to see for all $n \geq 0$ that $\mathbf{y(n) = a^n}$

Since the response of the system for n < 0 is defined to be zero, the unit sample response becomes

$$\mathbf{h(n) = a^n\, u(n)}$$

Example

Given the first order linear constant coefficient difference equation

$$y(n) = a\ y(n-1) + x(n)$$

find the impulse response of the system by first finding the system function H(z) and then taking the inverse transform. Assuming the system is causal.

Solution:

$y(n) = a\ y(n-1) + x(n)$

$Y(z) = a\ z^{-1}Y(z) + X(z) \quad ; \quad Y(z)(1 - a\ z^{-1}) = X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-a\ z^{-1}} = \frac{z}{z-a}$$

By the causality requirement, the inverse transform of H(z) is a positive time exponential and with the ROC z > a, then, h(n) becomes

$$\mathbf{h(n) = z^{-1}\{\frac{z}{z-a}\} = a^n\ u(n)}$$

Transfer Function (System Function)

The output y(n) of LTI system to an input sequence x(n) can be obtained by computing the convolution of x(n) with the unit sample (Impulse) response h(n) of the system.

$$\mathbf{y(n) = h(n) * x(n)}$$

Expressing this relationship in the z-domain as

$$\mathbf{Y(z) = H(z)\ X(z)}$$

Impulse Response:

$$\boldsymbol{H(z) = \frac{Y(z)}{X(z)}}$$

The LTI system can be described by means of a constant coefficient linear difference equation as follows:

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

$$Y(z) = \sum_{k=0}^{N} b(k)z^{-k}X(z) - \sum_{k=1}^{M} a(k)z^{-k}Y(z)$$

The transfer function of the LTI system,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{1 + \sum_{k=1}^{M} a(k)z^{-k}}$$

Example

Determine the transfer function and the impulse response of the following system:

y(n) = ½ y(n-1) +2x(n)

Solution:

Z-transform $\mathbf{Y(z) = \tfrac{1}{2}\, z^{-1}\, Y(z) + 2X(z)}$

Transfer function $\mathbf{\frac{Y(z)}{X(z)} = H(z) = \frac{2}{1-\frac{1}{2}z^{-1}}}$

The inverse transform $\mathbf{h(n) = 2(\tfrac{1}{2})^{n}\, u(n)}$

Example

Find the response of the system **y(n) = x(n+1) + x(n) + x(n-1)**, then find the system function in z-domain.

Solution:

y(n) = x(n+1) + x(n) + x(n-1)

Y(z) = X(z) z^{1} + X(z) + X(z) z^{-1}

$\mathbf{Y(z) = X(z)\,[z^{1} + 1 + z^{-1}]}$ **system response**

$\mathbf{H(z) = \frac{Y(z)}{X(z)} = z^{1} + 1 + z^{-1}}$ **system function**

Example

Find the difference equation realization and the frequency response of the system represented by the following causal system function H(z) = $\frac{(z+1)}{(z^{2}-2z+3)}$

Solution:

H(z) = $\frac{Y(z)}{X(z)} = \frac{(z+1)}{(z^2-2z+3)} = \frac{z^{-1}+z^{-2}}{1-2z^{-1}+3z^{-2}}$; now cross multiply

Y(z)$(1-2z^{-1}+3z^{-2})$ = X(z)$(z^{-1}+z^{-2})$

By taking the z^{-1} yields the difference equation

y(n) – 2y(n-1) + 3y(n-2) = x(n-1) + x(n-2)

y(n) = 2y(n-1) - 3y(n-2) + x(n-1) + x(n-2)

The freq. response H($e^{j\omega}$) obtained by letting z=$e^{j\omega}$

H($\boldsymbol{e^{j\omega}}$) = $\frac{(z+1)}{(z^2-2z+3)} = \frac{\boldsymbol{e^{j\omega}+1}}{\boldsymbol{e^{2j\omega}-2e^{j\omega}+3}}$