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DYNAMICS AND ENERGETICS OF A STRING VIBRATING AGAINST AN OBSTACLE PLACED AT ONE BOUNDARY

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ABSTRACT

In this paper we investigate the vibration of a string with a rigid obstacle placed at one of its boundaries. During vibration, a portion of the string wraps and unwraps around the obstacle. The impact between the string and the obstacle during wrapping is assumed to be inelastic. The length of the string that wraps around the obstacle, is discretized into a finite number of segments for the purpose of analysis. These segments sequentially collide with the obstacle starting from when the string makes first contact with the obstacle till it comes to rest. During wrapping, the energy of the string is dissipated through impact but during unwrapping the energy is conserved. The geometry of the string at any instant of time is determined from the boundary conditions associated with wrapping and unwrapping of the string. A general solution for vibration against convex obstacles of arbitrary geometry was analysed and numerical simulation results are presented for elliptic- and circular-shaped obstacles with different orientations and for different modes of string vibration. The results show that an obstacle at the boundary can be used as a passive mechanism for vibration suppression. The energy dissipated is found to be greater for higher modes of vibration and for obstacle geometries that result in greater length of wrapping.

INTRODUCTION

The dynamics of strings vibrating against obstacles first appeared in the technical literature with the work by Citrini [1]. Citrini examined point-shaped obstacles. The element of string that comes in contact with such obstacles can be assumed to be massless, and therefore the energy of the string, in the absence of damping, was assumed to remain conserved. Amerio [2] investigated the motion of a string vibrating against a rigid wall by modelling the unilateral constraint as a problem in impact. The nature of the impact was assumed to be elastic hence energy is conserving. Other researchers those work is based the premise of energy conservation include Schatzman [3] and Haraux and Cabannes [4]. In 1982, Burridge et al. [5] investigated the vibration of the sitar in which the bridge form a broad support, rather than a well-defined edge. During vibration, the sitar string wraps and unwraps around the gentle slope of the bridge and the length of the vibrating part of the string varies during oscillation [5]. Burridge et al. [5] modeled the impact of the string with the bridge as perfectly inelastic, discarding the assumption of energy conservation. Subsequently, Bamberger and Schatzman [6] proved the existence of solutions which do not conserve energy with arbitrary obstacles and Ahn [7] claimed energy loss of the string vibrating against flat obstacles. We investigate the vibration of a string against a convex obstacle located at its boundary. We assume the string to wrap and unwrap around the obstacle during each oscillation and the wrapping to be perfectly inelastic. Assuming that the string vibrates in a single mode at all times, it is shown that energy loss is higher for higher modes of oscillation. Although oscillation in a higher mode results in less wrapping, higher energy loss results from higher kinetic energy of the string

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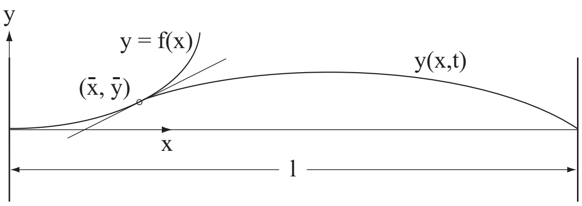

Figure 1. A STRING VIBRATING AGAINST AN OBSTACLE. IN WRAPPED CONFIGURATION, $(\bar{x}, \bar{y})$ DENOTES THE COORDINATE WHERE THE STRING BREAKS CONTACT WITH THE OBSTACLE.

during impact. The obstacle constrains the motion of the string and in this regard the mechanism for energy loss is a continuous-system version of the energy dissipation methodology proposed for finite degree-of-freedom systems by Issa, et al. [8]. Since the energy of the string decreases even in the absence of damping, the obstacle can be regarded as a passive mechanism for vibration suppression. A formal problem statement and a list of the assumptions made in our analysis is provided in section 2. In section 3 we present our analytical model for computing the geometry of the string as it wraps and unwraps around the obstacle. In section 4 we provide simulation results for percentage energy loss and length of wrapping during each cycle of oscillation for different modes. Section 6 provides concluding remarks.

Problem Statement and Assumptions

Consider a string vibrating against an obstacle placed at one of its boundaries, as shown in Fig.1. We made the following assumptions:

1. The obstacle is rigid and has the following geometry

$$y = f(x), \qquad y(0) = 0, \quad \left[\frac{dy}{dx}\right]_{x=0} = 0 \tag{1}$$

2. The string is homogenous and has a constant mass per unit length denoted by ρ. The tension in the string is equal to T and remains constant at all times. The string undergoes transverse vibration in the xy plane and is not affected by gravity.
3. The amplitude of oscillation of the string is small and therefore the equation of motion of the string can be expressed by the standard relation [11]

$$\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{1}{c^2}\left(\frac{\partial^2 y}{\partial t^2}\right), \qquad c \triangleq \sqrt{T/\rho} \tag{2}$$

   where $y(x,t)$ is the displacement of the string at a distance x from the origin at time t.
4. The string wraps around the obstacle during vibration. Over each time step during wrapping, a small element of the string comes to rest on the obstacle through perfectly inelastic collisions. The wrapping process continues till the freely vibrating portion of the string has no more kinetic energy.
5. The surface of the obstacle is not sticky and the string unwraps from the obstacle without any loss of energy.
6. At the initial time $t = 0$, the string has no contact with the obstacle. It is in its mean position with zero potential energy and kinetic energy equal to E_0.
7. The string continues to vibrate in the mode in which it started its vibration at the initial time. This implies that each point of the string, not in contact with the obstacle, has the same frequency of vibration[1] at any instant of time, and the number of nodes in the vibrating string remains constant.
8. The string has no internal damping, *i.e.*, the energy of the string will remain conserved during free vibration.

Analytical Model

Boundary conditions and general solution

A general solution to the partial differential equation in Eq.(2) can be written as [11]

$$y(x,t) = (\alpha_1 \sin\lambda x + \alpha_2 \cos\lambda x)\,(\alpha_3 \sin\omega t + \alpha_4 \cos\omega t) \tag{3}$$

where α_i, $i = 1,2,3,4$ are constants, ω is the circular frequency and λ is related to ω by the relation

$$\omega \triangleq c\lambda \tag{4}$$

At time $t = 0$, the string is at the mean position, *i.e.* $y(x,0) \equiv 0$, per assumption A6. Therefore, the solution in Eq.(3) is reduced to the form

$$y(x,t) = (A\sin\lambda x + B\cos\lambda x)\,\sin\omega t \tag{5}$$

At the right boundary, the string satisfies $y(l,t) = 0$ for all t. Thus Eq.(5) takes the form

$$y(x,t) = A\,(\sin\lambda x - \tan\lambda l\,\cos\lambda x)\,\sin\omega t \tag{6}$$

[1] The frequency of the string is not constant; it varies with time and amplitude when the string wraps and unwraps around the obstacle.

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We now consider the boundary conditions at the contact break point. From Fig.1 we have

$$f(\bar{x}) = y(\bar{x},t) \Longrightarrow f(\bar{x}) = A\left(\sin\lambda\bar{x} - \tan\lambda l\,\cos\lambda\bar{x}\right)\sin\omega t \quad (7)$$

Also, the string is tangential to the obstacle at the contact break point $x = \bar{x}$, *i.e.*,

$$f'(\bar{x}) = \frac{\partial y}{\partial x}(\bar{x},t) \Longrightarrow f'(\bar{x}) = \lambda A\left(\cos\lambda\bar{x} + \tan\lambda l\,\sin\lambda\bar{x}\right)\sin\omega t \quad (8)$$

From Eqs.(7) and (8) we get

$$\tan\lambda(l-\bar{x}) = -\lambda\frac{f(\bar{x})}{f'(\bar{x})} \quad (9)$$

which indicates that λ can be computed from the value of $\bar{x}$. The solution of Eq.(9) is however not unique - each non-trivial value of λ corresponds to a mode of vibration of the string. Since λ is an implicit function of $\bar{x}$, we can rewrite Eq.(8) as follows

$$A\sin\omega t = g(\bar{x}), \qquad g(x) \triangleq \frac{f'(x)}{\lambda\left(\cos\lambda x + \tan\lambda l\,\sin\lambda x\right)} \quad (10)$$

Equation (10) can be used to compute t from the value of $\bar{x}$. The existence of the solution, however, depends on the magnitude of A. We now discuss the procedure for computing A.
Let the total energy of the string at any time t be denoted by E. Then,

$$E = E_{\rm pe}^{\rm obs} + E_{\rm pe}^{\rm vib} + E_{\rm ke} \quad (11)$$

where $E_{\rm pe}^{\rm obs}$ is the potential energy of the string wrapped around the obstacle, $E_{\rm pe}^{\rm vib}$ is the potential energy of the freely vibrating string, and $E_{\rm ke}$ is the kinetic energy of the string. The total potential energy of the string is equal to the product of the tension T (which is assumed constant) and elongation of the string [12] and can be written as

$$\begin{aligned} E_{\rm pe}^{\rm obs} &= T\int_0^{\bar{x}}\left[\sqrt{1+[f'(x)]^2}-1\right]dx \\ E_{\rm pe}^{\rm vib} &= T\int_{\bar{x}}^{l}\left[\sqrt{1+[y'(x,t)]^2}-1\right]dx \\ &\approx \frac{T}{2}\int_{\bar{x}}^{l}\left[y'(x,t)\right]^2dx \Longrightarrow \end{aligned} \quad (12)$$

$$\begin{aligned} E_{\rm pe}^{\rm obs} &= \frac{T}{2}\lambda^2A^2\sin^2\omega t\int_{\bar{x}}^{l}\left[\cos\lambda x + \tan\lambda l\sin\lambda x\right]^2dx \\ &= \frac{1}{8}T\lambda A^2\sin^2\omega t\,\sec^2\lambda l\left\{2\lambda(l-\bar{x}) + \sin[2\lambda(l-\bar{x})]\right\} \\ &= \frac{1}{8}T\lambda\sec^2\lambda l\left\{2\lambda(l-\bar{x}) + \sin[2\lambda(l-\bar{x})]\right\}\left[g(\bar{x})\right]^2 \end{aligned} \quad (13)$$

$$\begin{aligned} E_{\rm ke} &= \frac{1}{2}\int_{\bar{x}}^{l}\rho\left[\dot{y}(x,t)\right]^2dx \\ &= \frac{\rho}{2}\omega^2A^2\cos^2\omega t\int_{\bar{x}}^{l}\left[\sin\lambda x - \tan\lambda l\cos\lambda x\right]^2dx \\ &= \frac{1}{8\lambda}\rho\,\omega^2A^2\cos^2\omega t\,\sec^2\lambda l \\ &\quad \left\{2\lambda(l-\bar{x}) - \sin[2\lambda(l-\bar{x})]\right\} \\ &= \frac{1}{8\lambda}\rho\,\omega^2\sec^2\lambda l\left\{2\lambda(l-\bar{x}) - \sin[2\lambda(l-\bar{x})]\right\} \\ &\quad \left\{A^2 - \left[g(\bar{x})\right]^2\right\} \end{aligned} \quad (14)$$

From Eqs.(11), (12), (13) and (14) it is easy to verify that the energy expression has the form

$$E = w(\bar{x}, A) \quad (15)$$

For a configuration in which the string is wrapped around the obstacle, the complete solution can be determined from the values of $\bar{x}$ and E using the four-step algorithm below:

1. Use Eq.(9) to determine the value of λ. Since Eq.(10) provides multiple non-trivial solutions that correspond to different modes of vibration, the solution corresponding to the initial mode of vibration should be chosen - see assumption A7.
2. Use Eq.(4) to compute ω.
3. Compute A from Eq.(15) using the values of $\bar{x}$, E, λ and ω.
4. Compute the time t from Eq.(10) by substituting in the values of $\bar{x}$, A and λ.

The complete solution can now be described using Eqs.(1) and (6) as follows:

$$y(x,t) = \begin{cases} f(x) & : x \in [0,\bar{x}] \\ A\left(\sin\lambda x - \tan\lambda l\,\cos\lambda x\right)\sin\omega t & : x \in [\bar{x}, l] \end{cases} \quad (16)$$

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Figure 2. SMALL STRING SEGMENT Δx WRAPS AROUND OBSTACLE AFTER PERFECTLY INELASTIC COLLISION. IN THE MGGNIFIED IMAGE, AB DENOTES THE SMALL STRING SEGMENT OF LENGTH Δx THAT WRAPS AROUND OBSTACLE OVER THE REGION AC.

Wrapping of the string

We discuss the method for computing the values of $\bar{x}$ and E at regular intervals of time. Let $\{\bar{x}_i, E_i\}$ denote the values of $\bar{x}$ and E at time $t = t_i$, $i = 0, 1, 2, \cdots, k$. We assume $t_0 = 0$. Then, from assumption A6, $\bar{x}_0 = 0$ and the value of E_0 is known. We will discuss the method for determining the value of t_k which denotes the time after which the string begins to unwrap. Let us assume that for some $i = j$, $\{\bar{x}_j, E_j\}$ is known. We outline the method for computing $\{\bar{x}_{j+1}, E_{j+1}\}$ from the values of $\{\bar{x}_j, E_j\}$. Choose a small segment of the vibrating string that is expected to wrap around the obstacle over a small interval of time. Let the projection of this string segment AB on the x axis be Δx as shown in Fig.2. The kinetic energy of this string segment, which will be lost due to inelastic collision, can be computed from Eq.(6) as follows

$$\begin{aligned} E_{\text{lost}} &= \frac{\rho}{2}\int_{\bar{x}_j}^{\bar{x}_j+\Delta x}\left[\dot{y}(x,t)\right]^2 dx \\ &= \frac{\rho}{2}\,\omega_j^2 A_j^2 \cos^2\omega_j t_j \int_{\bar{x}_j}^{\bar{x}_j+\Delta x}\left[\sin\lambda_j x - \tan\lambda_j l\,\cos\lambda_j x\right]^2 dx \end{aligned} \tag{17}$$

where A_j, ω_j, λ_j and t_j denote values of A, ω, λ and t, respectively, derived for the pair $\{\bar{x}_j, E_j\}$. Using Eq.(17), E_{j+1} can be computed as follows

$$E_{j+1} = E_j - E_{\text{lost}}, \qquad j = 1, 2, \cdots, k-1 \tag{18}$$

To compute $\bar{x}_{j+1}$, $j = 1, 2, \cdots, k-1$, we make the following general assumption:

A9. With reference to Fig.2, the potential energy of the vibrating string segment AB at time t_j is equal to the potential energy of the string segment AC wrapped on the obstacle at time t_{j+1}.

Using Eqs.(12) and (13) assumption A9 can be mathematically expressed as follows

$$\begin{aligned} \int_{\bar{x}_j}^{\bar{x}_{j+1}}\left[\sqrt{1+[f'(x)]^2}-1\right]dx &= \int_{\bar{x}_j}^{\bar{x}_j+\Delta x}\left[\sqrt{1+[y'(x,t)]^2}-1\right]dx \\ &\approx \frac{1}{2}\int_{\bar{x}_j}^{\bar{x}_j+\Delta x}\left[y'(x,t)\right]^2 dx \\ &= \frac{1}{2}\lambda_j^2 A_j^2 \sin^2\omega_j t_j\,\eta(\lambda_i, x_i) \end{aligned} \tag{19}$$

Where

$$\eta(\lambda_i, x_i) = \int_{\bar{x}_j}^{\bar{x}_j+\Delta x}\left[\cos\lambda_j x + \tan\lambda_j l\,\sin\lambda_j x\right]^2 dx$$

Equation (20) can be used to determine $\bar{x}_{j+1}$. The values of A_{j+1}, ω_{j+1}, λ_{j+1} and t_{j+1} are computed from the values of $\bar{x}_{j+1}$ and E_{j+1}. The iterative process is terminated when the kinetic energy of the vibrating string segment becomes approximately equal to zero. At this time, which is denoted as t_k, the string stops wrapping and begins to unwrap.

Unwrapping of the string

Similar to wrapping, the geometry of the string during unwrapping is computed from the values of $\bar{x}$ and E. The string begins to unwrap at $t = t_k$; at this time the values of $\bar{x} = \bar{x}_k$ and $E = E_k$ are known. Let $\{\bar{x}_i, E_i\}$ denote the values of $\bar{x}$ and E at time $t = t_i$, $i = k, k+1, k+2, \cdots, l$, where t_l denotes the time when the string has unwrapped completely. We outline the

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Table 1. PERCENTAGE ENERGY LOSS OVER ONE CYCLE OF OSCILLATION AND $\bar{x}_k$ FOR DIFFERENT VALUES OF E_0 AND THREE MODES OF OSCILLATION, ALL WITH $R = 1\,m$.

	Mode 1	Mode 2	Mode 3
$E_0 = 1.00\,J$	0.491%, 0.729 m	1.598%, 0.696 m	2.752%, 0.654 m
$E_0 = 0.50\,J$	0.256%, 0.596 m	0.861%, 0.569 m	1.547%, 0.537 m
$E_0 = 0.25\,J$	0.113%, 0.461 m	0.402%, 0.445 m	0.766%, 0.424 m

method for computing $\{\bar{x}_{j+1}, E_{j+1}\}$ from the values of $\{\bar{x}_j, E_j\}$ for $k \le j \le l-1$. One chooses a small segment of the string that is expected to unwrap over a small interval of time. Then we let the projection of this string segment on the x axis be Δx. Then,

$$\bar{x}_{j+1} = \bar{x}_j - \Delta x, \qquad j = k, k+1, \cdots, l-1 \tag{20}$$

Since there is no loss of kinetic energy during unwrapping (see assumption A5), we have

$$E_{j+1} = E_j, \qquad j = k, k+1, \cdots, l-1 \tag{21}$$

The values of A_{j+1}, ω_{j+1}, λ_{j+1} and t_{j+1} are computed iteratively from the values of $\bar{x}_{j+1}$ and E_{j+1}. The iterative process is terminated at $t = t_l$ when the potential energy of the string is equal to its value at the mean position.

Numerical Simulations

Consider a string with

$$T = 1\text{ N}, \qquad \rho = 0.025\text{ kg/m}, \qquad l = 4\text{ m} \tag{22}$$

The obstacle is assumed to be a circle of radius R and center coordinates $(x, y) \equiv (0, R)$, *i.e.*,

$$y = f(x) = R - \sqrt{R^2 - x^2}, \qquad 0 \le x \le R \tag{23}$$

It can be verified that $f(x)$ in Eq.(23) satisfies the boundary conditions in Eq.(1). For $R = 1$ m and $\Delta x = 0.001$ m, we compute the percentage loss of energy over one cycle of string oscillation for three different values of initial energy E_0 and for oscillation in the first, second, and third modes, respectively. These values are shown in Table 1 together with the values of $\bar{x}_k$, which is a measure of the length of wrapping around the obstacle. For the special case of $E_0 = 0.5\,J$, we plot the percentage loss of energy for three consecutive cycles of string vibration in the first two modes. These plots are shown in Fig.3. The following observations can be made from the plots in Figs.3 and data in Table 1:

For any mode of oscillation, it can be seen that the percentage energy loss is higher for higher values of E_0. This is not surprising since higher values of E_0 results in higher kinetic energy and greater length of wrapping, as evident from the values of $\bar{x}_k$ in Table 1, and consequently more energy loss through inelastic collision. The same argument can explain the reduction in the percentage loss of energy over consecutive cycles of vibration in Fig.3.

The percentage energy loss is higher for higher modes of oscillation for the same value of E_0. This is true for the same number of cycles (see Fig.3) as well as for the same length of time and is due to the fact that the velocities of the string associated with higher frequencies are higher in higher modes, and as a consequence the loss upon impact is higher. The value of $\bar{x}_k$ is less for the higher modes but this does not have a significant effect on the percentage of energy loss.

Table 2. PERCENTAGE ENERGY LOSS OVER ONE CYCLE OF OSCILLATION AND $\bar{x}_k$ FOR OBSTACLE OF DIFFERENT SHAPES AND SIZES, ALL WITH $E_0 = 0.50\,J$.

	Mode 1	Mode 2	Mode 3
Case (a)	0.029%, 0.295 m	0.113%, 0.291 m	0.237%, 0.286 m
Case (b)	0.256%, 0.596 m	0.861%, 0.569 m	1.547%, 0.537 m
Case (c)	0.915%, 0.897 m	2.549%, 0.815 m	3.887%, 0.737 m
Case (d)	0.665%, 0.803 m	2.021%, 0.750 m	3.278%, 0.692 m

(a) a circle with $R = 0.5$ m
(b) a circle with $R = 1.0$ m
(c) a circle with $R = 1.5$ m
(d) an ellipse with semi-major and semi-minor axes lengths of 1.2 m and 1.0 m, respectively, and with the major axis aligned with the x axis

and satisfy the boundary conditions in Eq.(1). The results are shown in Table 2. It is clear from the results that for circular obstacles the percentage energy loss increases with increase in radius and vice versa. This is in agreement with the results expected for the limiting cases, namely, percentage energy loss is zero when the radius of the circle is zero and is equal to 100% when the radius is infinity. The ellipse in case (d) circumscribes the circle in case (b) and provides a lower slope for the wrapping curve. A comparison of the data for cases (b) and (d) indicates that a slight decrease in slope of the obstacle results in significantly higher percentage of energy loss.

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Figure 3. PLOT OF PERCENTAGE ENERGY CONTENT OF THE STRING OVER THREE CONSECUTIVE CYCLES OF VIBRATION IN (a) MODE 1, AND (b) MODE 2. FOR BOTH CASES, THE INITIAL ENERGY OF THE STRING WAS $E_0 = 0.5\,J$.

Effect of Change in Slope of Obstacle

We consider the obstacle in Fig.4 where the curve $y = g(x)$ is obtained by rotating the curve $y = f(x)$ in Fig.1 clockwise by angle θ about point O. To deal with this problem, we modify assumptions A1 and A6 as follows:

A1. The obstacle is rigid and has the following geometry

$$y = f(x), \qquad y(0) = 0, \qquad \left[\frac{dy}{dx}\right]_{x=0} = -\tan\theta \tag{24}$$

A6. At the initial time $t = 0$, the string has no contact with the obstacle. It has zero kinetic energy and potential energy equal to E_0. The displacement of the string at the initial time corresponds to a single mode of free vibration as shown in Fig.4.

Figure 4. A string vibrating against an obstacle. The obstacle is identical to the one in Fig.1 but rotated clockwise by angle θ about point O.

The remaining assumptions, A2 through A5 and A6 through A9, are not changed. In Table 3 we present simulation results for a string with

$$T = 1\text{ N}, \qquad \rho = 0.025\text{ kg/m}, \qquad l = 4\text{ m}, \qquad E_0 = 0.50\text{ J} \tag{25}$$

and a circular obstacle of radius $R = 1$ m. A comparison of the results indicates that percentage energy loss is significantly higher for higher values of θ. In our analysis, the string was assumed to have no damping. As the string wraps around the obstacle, its effective length decreases and frequency of vibration increases - this will increase the rate of energy decay which depends on the product of damping ratio and natural frequency.

Conclusion

The vibration of a string wrapping and unwrapping around a smooth obstacle was investigated in this paper. Assuming linear behavior of the string, an analytical model was developed for computing its geometry at each time step by bookkeeping the energy. The energy of the string is assumed to dissipate during wrapping through inelastic collision between the string and the obstacle but remain conserved during unwrapping. The obstacle serves as a passive mechanism for damping and its effectiveness

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Table 3. PERCENTAGE ENERGY LOSS OVER ONE CYCLE OF OSCILLATION AND $\bar{x}_k$ FOR TWO MODES OF OSCILLATION WITH DIFFERENT VALUES θ.

θ	Mode 1	Mode 2
0 *deg*	0.256%, 0.596 m	0.113%, 0.291 m
15 *deg*	1.486%, 0.866 m	3.478%, 0.841 m
30 *deg*	9.233%, 1.095 m	13.72%, 1.080 m

can be increased by changing its orientation in a manner that results in greater wrapping. This leads to the possibility of rapid dissipation of vibration energy through active control of the orientation of the obstacle and extension of the methodology for active vibration control of soft structures such as thin plates and membranes.

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