AL FURAT AL AWSAT TECHNICAL UNIVERSITY

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DIGITAL SIGNAL PROCESSING

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Discrete-Time Fourier Transform (DTFT)

The discrete-time Fourier transform (DTFT) or simply the Fourier transform of a discrete time sequences x(n) is represented by the complex exponential sequence [$e^{-j\omega n}$] where ω is the real frequency variable. This transform is useful to map the time-domain sequence into a continuous function of a frequency variable. The DTFT and the z-transform are applicable to any arbitrary sequences, whereas the DFT can be applied only to finite length sequences.

The discrete-time Fourier transform $\mathbf{X(e^{j\omega})}$ of a sequence x(n) is defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

This equation represents the Fourier series representation of the periodic function $X(e^{j\omega})$. Hence, the Fourier coefficients x(n) can be determined from $X(e^{j\omega})$ using the Fourier integral expressed by

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$$

called the inverse discrete-time Fourier transform (IDTFT). The two equations above are known as the discrete-time Fourier transform pair for the sequence x(n), which relate the time and frequency domain.

Discrete Fourier Transform (DFT)

In time domain, representation of digital signals describes the signal amplitude versus the sampling time instant or the sample number. However, in some applications, signal frequency content is very useful than as digital signal samples.

The algorithm transforming the time domain signal samples to the frequency domain components is known as the discrete Fourier transform, or DFT. The DFT also establishes a relationship between the time domain representation and the frequency domain representation. Therefore, we can apply the DFT to perform frequency analysis of a time domain sequence.

The discrete Fourier transform (DFT) computes the values of the z-transform for evenly spaced points around the unit circle for a given sequence. If the sequence to be represented is of finite duration, i.e. has only a finite number of non-zero values, the transform used is discrete Fourier

transform (DFT). DFT finds its applications in digital signal processing including linear filtering, correlation analysis and spectrum analysis.

Definition

Let x(n) be a finite duration sequence. The N-point DFT of the sequence x(n) is expressed by

$$\boldsymbol{X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}}\ , \quad \boldsymbol{k = 0, 1, 2, \dots, N-1}$$

and the corresponding IDFT is

$$\boldsymbol{x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(k) e^{j2\pi nk/N}}\ , \quad \boldsymbol{n = 0, 1, 2, \dots, N-1}$$

Example

Find the DTFT of the following finite duration sequence of length L.

$$x(n) = \begin{cases} A\ , & 0 \le n \le L-1 \\ o\ , & otherwise \end{cases}$$

Also, find the inverse DTFT to verify x(n) for L = 3 and A = 1V.

Solution:

Taking the DTFT of the sequence, we get:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=0}^{L-1} A\, e^{-j\omega n}$$

$$= A \sum_{n=0}^{L-1} e^{-j\omega n} = A\left[\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right] = A\left[\frac{e^{\frac{j\omega L}{2}} . e^{-\frac{j\omega L}{2}} - e^{-\frac{j\omega L}{2}} . e^{-\frac{j\omega L}{2}}}{e^{\frac{j\omega}{2}} . e^{-\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}} . e^{-\frac{j\omega}{2}}}\right]$$

$$= A\left[\frac{(e^{-\frac{j\omega L}{2}})\,[\,e^{\frac{j\omega L}{2}} - e^{-\frac{j\omega L}{2}}]}{(e^{-\frac{j\omega}{2}})\,[\,e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}]}\right] = A\, e^{-\frac{j\omega(L-1)}{2}} \cdot \left[\frac{\sin(\frac{\omega L}{2})}{\sin(\frac{\omega}{2})}\right]$$

The DTFT of x(n) with L = 3 and A = 1V is given by

$$X(e^{j\omega}) = A \sum_{n=0}^{L-1} e^{-j\omega n} = 1 + e^{-j\omega} + e^{-j2\omega}$$

The inverse IDTFT of $X(e^{j\omega})$ is

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1 + e^{-j\omega} + e^{-j2\omega}\right)e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{jn\omega}}{jn} + \frac{e^{j(n-1)\omega}}{j(n-1)} + \frac{e^{j(n-2)\omega}}{j(n-2)}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{2\sin n\pi}{n} + \frac{2\sin((n-1)\pi)}{(n-1)} + \frac{2\sin((n-2)\pi)}{(n-2)}\right]$$

For n = 0

$$x(0) = \frac{1}{2\pi}\left[\frac{2\sin n\pi}{n}\right] = \frac{\sin n\pi}{n\pi} = 1 \;, \quad [L \text{ Hospital's rule gives } \lim_{n\to 0}\frac{\sin n\pi}{n\pi} = 1]$$

Similarly,

For n=1: x(1) =1; For n=2: x(2) =1; For n ≥ 3: x(n) =0

These values are identical to the defined sequence for L = 3 and A= lV.

i.e. $x(n) = \begin{cases} 1\,, & 0 \le n \le 2 \\ o\,, & otherwise \end{cases}$

<u>Example</u>

Determine the DFT of the sequence

$x(n) = \begin{cases} \frac{1}{4}\,, & 0 \le n \le 2 \\ o\,, & otherwise \end{cases}$

<u>Solution:</u>

The N-point DFT of the sequence x(n) is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}\,, \quad k = 0, 1, 2, \dots, N-1\,, \qquad x(n) = (\,¼\,,\,¼\,,\,¼\,)$$

Therefore,

$$X(k) = \frac{1}{4}\sum_{n=0}^{2} e^{-j2\pi nk/N} = \frac{1}{4}\left(1 + e^{-j\omega} + e^{-j2\omega}\right)\Big|_{\omega=2\pi k/N} = \frac{1}{4}e^{-j\omega}[1 + 2cos\omega]$$

Where $\omega = 2\pi k/N$ and N =3

$$X(k) = \frac{1}{4}e^{-j2\pi k/3}[1 + 2\cos(\frac{2\pi k}{3})\,]\text{ , where } k = 0, 1, 2, \dots$$

Relationship of the DFT to the Fourier series coefficients

The Fourier series of a periodic sequence $x_p(n)$ with fundamental period N is given by

$$x_p(n) = \sum_{n=0}^{N-1} c_k e^{j2\pi nk/N} \ , -\infty < n < \infty \ ,$$

Where the Fourier series coefficients are given by

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi nk/N} \ , \ \ k = 0, 1, 2, .., N-1$$

By comparing the above equations with that of DFT pair and defining a sequence x(n) which is identical to x_p (n) over a single period, we get

$$\boldsymbol{X(k) = Nc_k}$$

If a periodic sequence $x_p(n)$ is formed by periodically repeating x(n) every N samples, i.e.

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

The discrete frequency-domain representation is given by

$$X(k) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi nk/N} = Nc_k \ , \ \ k=0,\ 1,\ \ldots,\ N-1$$

And the IDFT is

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi nk}{N}} \ , \qquad -\infty < n < \infty$$

Relationship of the DFT to the z-transform

Let X(z) be the z-transform for a sequence x(n) which is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n).z^{-n}$$

with a ROC that includes the unit circle. If X(z) is sampled at the N equally spaced points on the unit circle,

$z_k = e^{\frac{j2\pi nk}{N}}$, $k = 0,\ 1, \ldots,\ N-1$, then

$X(k) = X(z)|_{z=e^{j2\pi k/N}}$, $k = 0,\ 1, \ldots,\ N-1$

$$= \sum_{n=-\infty}^{\infty} x(n).e^{-j2\pi k/N}$$

This is identical to the Fourier transform $X(e^{j\omega})$ evaluated at the N equally spaced frequencies $\boldsymbol{\omega_k = 2\pi k/N,\ k = 0, 1, \ldots, N-1}$

If the sequence r(n) has a finite duration of length N, then the z-transform is given by

$$\boldsymbol{X(z) = \sum_{n=0}^{N-1} x(n).z^{-n}}$$

Substituting the IDFT relation for x(n), we get

$$\boldsymbol{X(z)} = \sum_{\boldsymbol{n=0}}^{\boldsymbol{N-1}} \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{\frac{j2\pi nk}{N}}\right].\boldsymbol{z^{-n}} = \frac{1}{N}\sum_{\boldsymbol{k=0}}^{\boldsymbol{N-1}} X(k) \sum_{n=0}^{N-1}\left(e^{\frac{j2\pi k}{N}}.z^{-1}\right)^{n}$$

$$= \frac{1}{N}\sum_{\boldsymbol{k=0}}^{\boldsymbol{N-1}} X(k).\frac{1-z^{-N}}{1-e^{\frac{j2\pi k}{N}}.z^{-1}} = \boldsymbol{\frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{\frac{j2\pi k}{N}}.z^{-1}}}$$

The above equation is identical to that of frequency sampling form. When this is evaluated over a unit circle, then

$$\boldsymbol{X(e^{j\omega}) = \frac{1-e^{-j\omega N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{j(\omega-2\pi k/N)}}}$$

Properties of the DFT

The properties of the DFT are useful in the practical techniques for processing signals. The various properties are given below.

I. Periodicity

If *X(k)* is an N-point DFT of *x(n)*, then

$$\boldsymbol{x(n + N) = x(n) \text{ for all } n}$$

$$\boldsymbol{X(k + N) = X(k) \text{ for all } k}$$

II. Linearity

If $X_1(k)$ and $X_2(k)$ are the N-point DFTs of $x_1(n)$ and $x_2(n)$ respectively, a and b are arbitrary constants either real or complex-valued, then

$$ax_1(n) + bx_2(n) \overset{DFT}{\longleftrightarrow}_N aX_1(k) + bX_2(k)$$

III. Time Reversal of a Sequence

If $x(n) \overset{DFT}{\longleftrightarrow}_N X(k)$ then,

$$x(N - n) \overset{DFT}{\longleftrightarrow}_N X(N - k)$$

Hence, when the *N*-point sequence in time is reversed, it is equivalent to reversing the DFT values.

Example

Find the 4-Point DFT of the sequence $x(n) = \cos\frac{n\pi}{4}$.

Solution:

Given N =4

$x(n) = \{\cos(0), \cos(\frac{\pi}{4}), \cos(\frac{\pi}{2}), \cos(\frac{3\pi}{4})\}$ = **{1, 0.707, 0, -0.707}**

The N-point DFT of the sequence x(n) is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}, \quad k = 0, 1, 2, \dots, N-1$$

$X(k) = \sum_{n=0}^{3} x(n)e^{-j2\pi nk/4}, \quad k = 0, 1, 2, 3$

$= \sum_{n=0}^{3} x(n)e^{-j\pi nk/2}, k = 0, 1, 2, 3$

For k = 0

$X(0) = = \sum_{n=0}^{3} x(n)e^{-j\pi(0)n/2} = \sum_{n=0}^{3} x(n) = \mathbf{1}$

For k = 1

$X(1) = \sum_{n=0}^{3} x(n)e^{-j\pi\,(1)n/2} = 1 + 0.707e^{-j\pi/2} + 0 + (-0.707)e^{-j3\pi/2}$

$$= 1 + 0.707\,(-j) + 0 - (0.707)(j) = \mathbf{1 - j\,1.414}$$

For k = 2

$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi\,(2)n/2} = 1 + 0.707e^{-j\pi} + 0 + (-0.707)e^{-j3\pi}$

$$= 1 + 0.707\,(-1) + 0 - (0.707)(-1) = \mathbf{1}$$

For k = 3

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j\pi\,(3)n/2} = 1 + 0.707e^{-j3\pi/2} + 0 + (-0.707)e^{-j9\pi/2}$$

$$= 1 + 0.707\,(j) + 0 - (0.707)(-j) = \mathbf{1 + j\,1.414}$$

$$\boldsymbol{X(k) = \{1,\ 1 - j\,1.414,\ 1,\ 1 + j\,1.414\}}$$

<u>Example</u>

Find the inverse DFT of $X(k) = \{1, 2, 3, 4\}$.

<u>Solution:</u>

The IDFT is given by

$$\boldsymbol{x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X(k)e^{j2\pi nk/N}\ ,\quad n = 0, 1, 2, \ldots, N-1}$$

Given N = 4, $x(n) = \frac{1}{4}\sum_{n=0}^{3} X(k)e^{\,j2\pi nk/4}\ ,\quad n = 0, 1, 2, 3$

When n = 0

$x(0) = \frac{1}{4}\sum_{n=0}^{3} X(k)e^{\,j2\pi(0)k/4} = \frac{1}{4}(1 + 2 + 3 + 4) = \mathbf{5/2}$

When n = 1

$X(1) = \frac{1}{4}\sum_{n=0}^{3} X(k)e^{\,j2\pi(1)k/4} = \frac{1}{4}\left(1 + 2e^{j\pi/2} + 3e^{j\pi} + 4e^{j3\pi/2}\right)$

$$= \frac{1}{4}(1 + 2(j) + 3(-1) + 4(-j)) = \frac{1}{4}\,(2 - 2j) = \mathbf{\frac{1}{2} - \frac{1}{2}\,j}$$

When n = 2

$X(2) = \frac{1}{4}\sum_{n=0}^{3} X(k)e^{\,j2\pi(2)k/4} = \frac{1}{4}\left(1 + 2e^{j\pi} + 3e^{j2\pi} + 4e^{j3\pi}\right)$

$$= \frac{1}{4}(1 + 2(-1) + 3(1) + 4(-1)) = \frac{1}{4}\,(-2) = \mathbf{-\frac{1}{2}}$$

When n = 3

$X(3) = \frac{1}{4}\sum_{n=0}^{3} X(k)e^{\,j2\pi(3)k/4} = \frac{1}{4}\left(1 + 2e^{j3\pi/2} + 3e^{j3\pi} + 4e^{j9\pi/2}\right)$

$$= \frac{1}{4}(1 + 2(-j) + 3(-1) + 4(j)) = \ -\mathbf{\frac{1}{2}} + \mathbf{\frac{1}{2}}\boldsymbol{j}$$

$$\mathbf{X(n) = \{\frac{5}{2}, \frac{1}{2} - \frac{1}{2}\ j, -\frac{1}{2}, -\frac{1}{2} + \frac{1}{2} j\}}$$