AL FURAT AL AWSAT TECHNICAL UNIVERSITY

NAJAF COLLEGE OF TECHNOLOGY

DEPARTMENT OF TECHNICAL COMMUNICATIONS ENGINEERING

DIGITAL SIGNAL PROCESSING

3rd YEAR

Academic Responsible

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2014/2015

Amplitude Spectrum and Power Spectrum

One of the DFT applications is transformation of a finite-length digital signal x(n) into the spectrum in frequency domain. Figure below demonstrates such an application, where A_k and P_k are the computed amplitude spectrum and the power spectrum, respectively, using the DFT coefficients X(k).

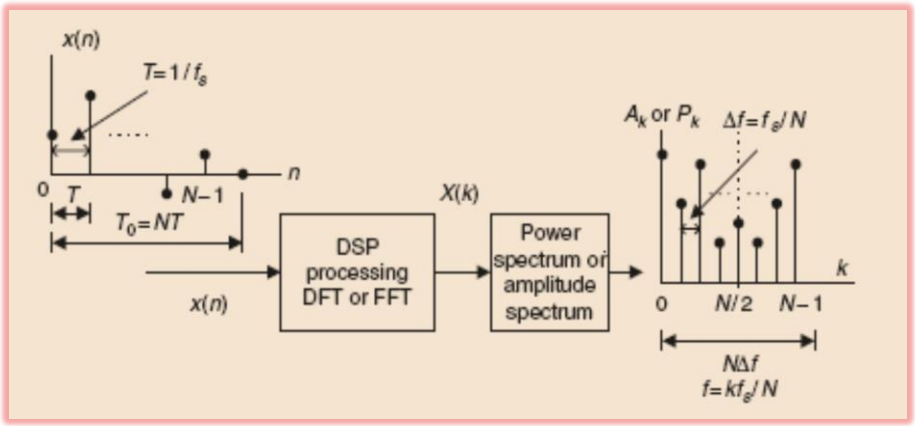


Figure 5.1 Applications of DFT/ FFT

First, we achieve the digital sequence x(n) by sampling the analog signal x(t) and truncating the sampled signal with a data window with a length T_0 = NT, where T is the sampling period and N the number of data points. The time for data window is T_0 = NT.

Next, we apply the DFT to the obtained sequence x(n), to get the N-DFT coefficients

$$\boldsymbol{X(k)} = \sum_{n=0}^{N-1} \boldsymbol{x(n)e^{-j2\pi nk/N}}\ , \quad \boldsymbol{k = 0, 1, 2, \dots, N-1}$$

We define the amplitude spectrum as:

$$\boldsymbol{A_k = \frac{1}{N}|X(k)| = \frac{1}{N}\sqrt{(Real[X(k)])^2 + (Imag[X(k)])^2}}\ , \boldsymbol{k = 0, 1, 2, \dots, N-1}$$

Keeping original DC term at k = 0, a one-sided amplitude spectrum for equation above is:

$$\boldsymbol{\overline{A}_k = \begin{cases} \frac{1}{N}|X(0)|, & k=0 \\ \frac{2}{N}|X(k)|, & k=1,2,\dots,\frac{N}{2} \end{cases}}$$

Correspondingly, the phase spectrum is given by:

$$\boldsymbol{\varphi_k = tan^{-1}\left(\frac{Imag[X(k)]}{Real[X(k)]}\right), \qquad k = 0, 1, 2, \dots, N-1}$$

Besides the amplitude spectrum, the power spectrum is also used. The DFT power spectrum is defined as:

$$\boldsymbol{P_k = \frac{1}{N^2}|X(k)|^2 = \frac{1}{N^2}\{(Real[X(k)])^2 + (Imag[X(k)])^2\}, k = 0, 1, 2, \dots, N-1}$$

Similarly, for a one-sided power spectrum, we get:

$$\boldsymbol{\overline{P}_k = \begin{cases} \frac{1}{N^2}|X(0)|^2, & k=0 \\ \frac{2}{N^2}|X(k)|^2, & k=1,2,\dots,\frac{N}{2} \end{cases}}$$

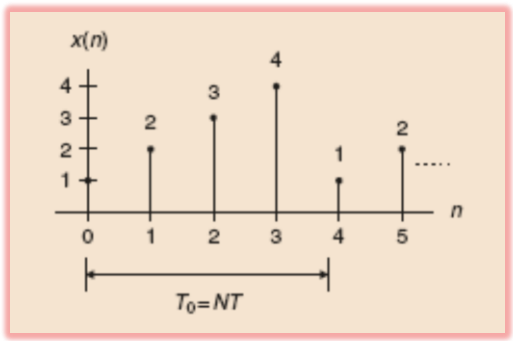
And $$\boldsymbol{f = \frac{kf_s}{N}}$$

The frequency resolution is defined in the equation $\Delta f = f_s/N$. It follows that better frequency resolution can be achieved by using a longer data sequence.

<u>Note</u>: *We can define the frequency resolution as the frequency step between two consecutive DFT coefficients to measure how fine the frequency domain presentation is.*

<u>Example</u>

Consider the sequence:



Assuming that fs = 100 Hz. Compute the amplitude spectrum, phase spectrum and power spectrum.

<u>Solution:</u>

Since N = 4, DFT coefficients are: X(0) = 10,

X(1) = −2 + j 2 , X(2) = −2, X(3) = −2 −j 2

For k = 0, $f = k\, f_s/N$ = 0*100/4 = 0Hz

$A_0 = \frac{1}{4}|X(0)| = 2.5$

$P_0 = \frac{1}{4^2}|X(0)|^2 = 6.25$

$$\varphi_0 = \tan^{-1}\left(\frac{\text{Imag}[X(0)]}{\text{Real}[X(0)]}\right) = 0^0$$

Similarly:

K	f	A_K	Φ_K *in degree*	P_K
1	25	0.7071	135	0.5
2	50	0.5	180	0.25
3	75	0.7071	-135	0.5

We can easily find the one-sided amplitude spectrum and one-sided power spectrum as:

$\bar{A}_0 = 2.5,\ \bar{A}_1 = 1.4141,\ \bar{A}_2 = 1$ and
$\bar{P}_0 = 6.25,\ \bar{P}_1 = 2,\ \bar{P}_2 = 1.$

Window Functions

Spectral Estimation Using Window Functions

Consider the pure 1-Hz sine wave with 32 samples shown in the figure below.

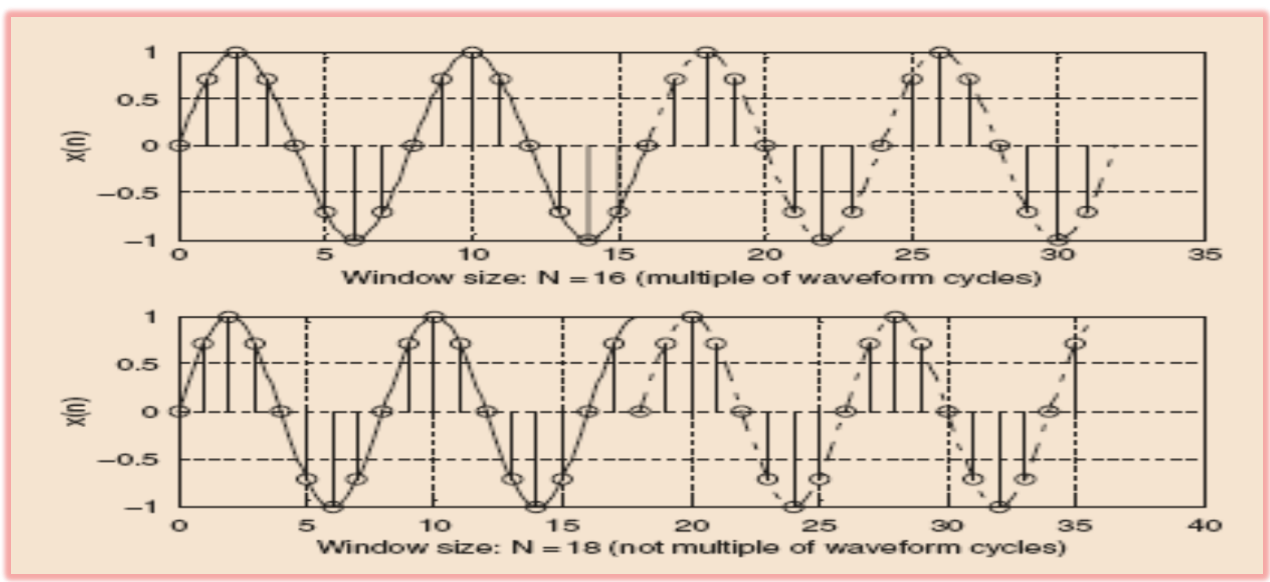


Figure 5.2 Sampling a 1-Hz sine wave using (top) 16 samples per cycle and (bottom) 18 samples per cycle.

As shown in the figure, if we use a window size of N =16 samples, which is a multiple of the two waveform cycles, the second window repeats with continuity. However, when the window size is chosen to be 18 samples, which is not a multiple of the waveform cycles (2.25 cycles), the second window repeats the first window with discontinuity. *This discontinuity produces harmonic frequencies that are not present in the original signal (spectral leakage).* Figure below shows the spectral plots for both cases using the DFT/FFT directly.

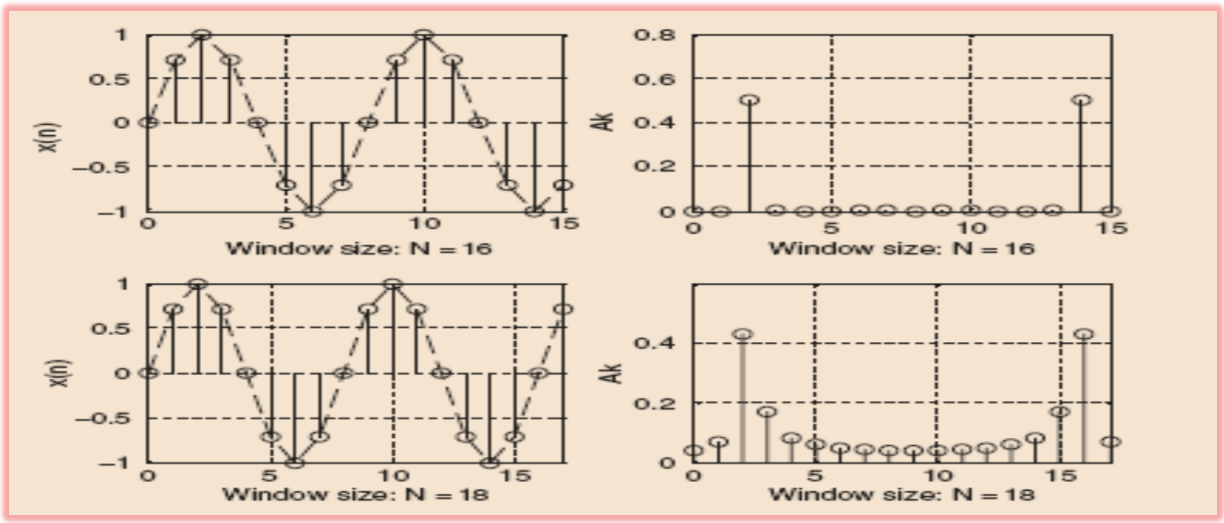


Figure 5.3 Signal samples and spectra without and with spectral leakage.

The amount of spectral leakage shown in the second plot is due to amplitude discontinuity in time domain. The bigger the discontinuity, the more is the leakage. To reduce the effect of spectral leakage, a window function can be used whose amplitude tapers smoothly and gradually toward zero at both ends. Applying the window function w(n) to a data sequence x(n) to obtain a windowed sequence $x_w(n)$ is better illustrated in the figure below using:

$$\boldsymbol{x_w(n) = x(n)w(n), for\ n = 0, 1, \dots, N-1}$$

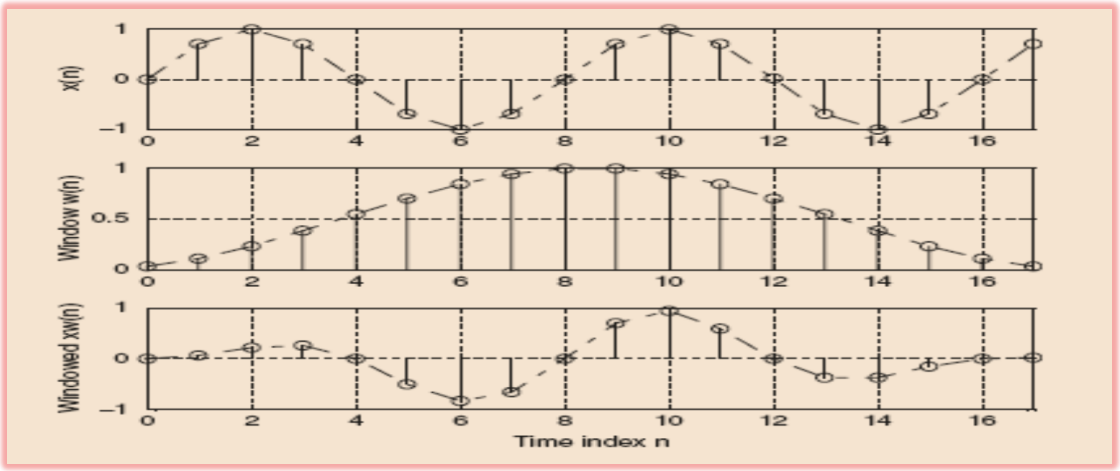


Figure 5.4 Illustration of the window operation.

Types of window functions (common functions)

1- **The rectangular window (no window function):**

$$w_R(n) = 1 \qquad for \quad 0 \le n \le N-1$$

2- **The triangular window:**

$$w_{tri}(n) = 1 - \frac{|2n - N + 1|}{N-1} \qquad for \quad 0 \le n \le N-1$$

3- **The Hamming window:**

$$w_{hm}(n) = 0.54 - 0.46\, cos\left(\frac{2\pi n}{N-1}\right) \qquad for \quad 0 \le n \le N-1$$

4- **The Hanning window:**

$$w_{hn}(n) = 0.5 - 0.5\, cos\left(\frac{2\pi n}{N-1}\right) \qquad for \quad 0 \le n \le N-1$$

Plots for each window function for a size of 20 samples are shown in Figure below:

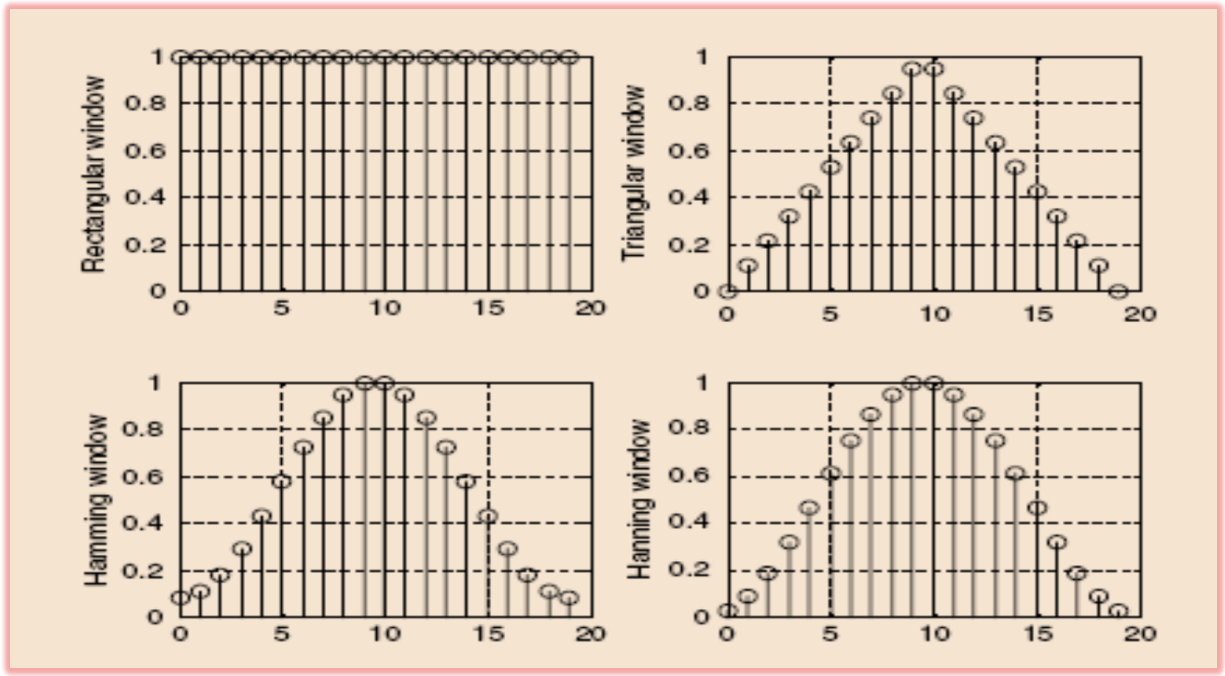


Figure 5.5 Plots of window sequences

Example

Consider the sequence x(0) = 1, x(1) = 2, x(2) = 3 and x(3) = 4, and given fs = 100 Hz, T = 0.01 seconds, compute the amplitude spectrum, phase spectrum and power spectrum

a. Using the triangular window function.

b. Using the Hamming window function.

Solution:

a) Since N = 4, from the triangular window function equation, we have:

$w_{tri}(0) = 0$, $w_{tri}(1) = 0.6667$, $w_{tri}(2) = 0.6667$, and $w_{tri}(3) = 0$.

Now, by applying equation $\mathbf{x_w(n) = x(n)w(n), for\ n = 0, 1, \ldots, N-1}$, we have:

$x_w(0) = x(0)\, w_{tri}(0) = 0$. Similarly $x_w(1) = 1.3334$, $x_w(2) = 2$, and $x_w(3) = 0$

Applying DFT equation to $x_w(n)$ for K=0, 1, 2 and 3, we have:

$X(0) = 3.3334$, $X(1) = -2 - j1.3334$, $X(2) = 0.6666$ and $X(3) = -2 + j\,1.3334$

$\Delta f = 1 / NT = 25$ Hz

Applying equations of amplitude spectrum, phase spectrum and power spectrum, we get

$A_0 = \frac{1}{4}|X(0)| = 0.8334$

$P_0 = \frac{1}{4^2}|X(0)|^2 = 0.6954$

$\varphi_0 = \tan^{-1}\left(\frac{0}{3.3334}\right) = 0^0$

Similarly:

K	A_K	Φ_K *in degree*	P_K
1	0.6009	− 146.31	0.3611
2	0.1667	0	0.0278
3	0.6009	146.31	0.3611

b) Since N = 4, from the Hamming window function, we have:

$w_{hm}(0) = 0.08$, $w_{hm}(1) = 0.77$, $w_{hm}(2) = 0.77$ and $w_{hm}(3) = 0.08$. The windowed sequence is computed using $\mathbf{x_w(n) = x(n)w(n)}$ as:

$x_w(0) = x(0)\, w_{hm}(0) = 0.08$, $x_w(1) = 1.54$, $x_w(2) = 2.31$ and $x_w(3) = 0.32$

Applying DFT equation to $x_w(n)$ for K=0, 1, 2 and 3, we have:

X(0)= 4.25, X(1) = −2.23 − j1.22, X(2) = 0.53 and X(3) = − 2.23 + j 1.22

$\Delta f = 1 / NT = 25$ Hz

Applying equations of amplitude spectrum, phase spectrum and power spectrum, we get

$A_0 = \frac{1}{4}|X(0)| = 1.0625$

$P_0 = \frac{1}{4^2}|X(0)|^2 = 1.1289$

$\varphi_0 = \tan^{-1}\left(\frac{0}{4.25}\right) = 0^0$

Similarly:

K	A_K	Φ_K *in degree*	P_K
1	0.6355	−151.32	0.4308
2	0.1325	0	0.0176
3	0.6355	151.32	0.4308