AL FURAT AL AWSAT TECHNICAL UNIVERSITY
NAJAF COLLEGE OF TECHNOLOGY
DEPARTMENT OF TECHNICAL COMMUNICATIONS ENGINEERING

**DIGITAL SIGNAL PROCESSING**

3rd YEAR

Academic Responsible

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Amplitude Spectrum and Power Spectrum

One of the DFT applications is transformation of a finite-length digital signal \(x(n)\) into the spectrum in frequency domain. Figure below demonstrates such an application, where \(A_k\) and \(P_k\) are the computed amplitude spectrum and the power spectrum, respectively, using the DFT coefficients \(X(k)\).

First, we achieve the digital sequence \(x(n)\) by sampling the analog signal \(x(t)\) and truncating the sampled signal with a data window with a length \(T_0 = NT\), where \(T\) is the sampling period and \(N\) the number of data points. The time for data window is \(T_0 = NT\).

Next, we apply the DFT to the obtained sequence \(x(n)\), to get the \(N\)-DFT coefficients

\[
X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}, \quad k = 0, 1, 2, ..., N - 1
\]

We define the amplitude spectrum as:

\[
A_k = \frac{1}{N} |X(k)| = \frac{1}{N} \sqrt{(Real[X(k)])^2 + (Imag[X(k)])^2}, k = 0, 1, 2, ..., N - 1
\]

Keeping original DC term at \(k = 0\), a one-sided amplitude spectrum for equation above is:

\[
\overline{A}_k = \begin{cases} 
\frac{1}{N}|X(0)|, & k=0 \\
\frac{2}{N}|X(k)|, & k=1,2,...,\frac{N}{2}
\end{cases}
\]

Correspondingly, the phase spectrum is given by:
Besides the amplitude spectrum, the power spectrum is also used. The DFT power spectrum is defined as:

\[ P_k = \frac{1}{N^2} |X(k)|^2 = \frac{1}{N^2} \left( (\text{Real}[X(k)])^2 + (\text{Imag}[X(k)])^2 \right), \quad k = 0, 1, 2, \ldots, N - 1 \]

Similarly, for a one-sided power spectrum, we get:

\[ \overline{P}_k = \begin{cases} \frac{1}{N^2} |X(0)|^2, & k = 0 \\ \frac{2}{N^2} |X(k)|^2, & k = 1, 2, \ldots, \frac{N}{2} \end{cases} \]

And

\[ f = \frac{KF_s}{N} \]

The frequency resolution is defined in the equation \( \Delta f = f_s / N \). It follows that better frequency resolution can be achieved by using a longer data sequence.

**Note:** We can define the frequency resolution as the frequency step between two consecutive DFT coefficients to measure how fine the frequency domain presentation is.

**Example**

Consider the sequence:

Assuming that \( f_s = 100 \) Hz. Compute the amplitude spectrum, phase spectrum and power spectrum.

**Solution:**

Since \( N = 4 \), DFT coefficients are: \( X(0) = 10 \),

\( X(1) = -2 + j \ 2 \), \( X(2) = -2 \), \( X(3) = -2 - j \ 2 \)

For \( k = 0, f = Kf_s/N = 0*100/4 = 0H \)

\[ A_0 = \frac{1}{4} |X(0)| = 2.5 \]
\[ P_0 = \frac{1}{4^2} |X(0)|^2 = 6.25 \]

\[ \varphi_0 = \tan^{-1} \left( \frac{\text{Imag}[X(0)]}{\text{Real}[X(0)]} \right) = 0^\circ \]

Similarly:

<table>
<thead>
<tr>
<th>K</th>
<th>( f )</th>
<th>( A_K )</th>
<th>( \Phi_K \text{ in degree} )</th>
<th>( P_K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td>0.7071</td>
<td>135</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>0.5</td>
<td>180</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>75</td>
<td>0.7071</td>
<td>-135</td>
<td>0.5</td>
</tr>
</tbody>
</table>

We can easily find the one-sided amplitude spectrum and one-sided power spectrum as:

\[ \tilde{A}_0 = 2.5, \tilde{A}_1 = 1.4141, \tilde{A}_2 = 1 \text{ and } \tilde{P}_0 = 6.25, \tilde{P}_1 = 2, \tilde{P}_2 = 1. \]

**Window Functions**

**Spectral Estimation Using Window Functions**

Consider the pure 1-Hz sine wave with 32 samples shown in the figure below.
As shown in the figure, if we use a window size of N = 16 samples, which is a multiple of the two waveform cycles, the second window repeats with continuity. However, when the window size is chosen to be 18 samples, which is not a multiple of the waveform cycles (2.25 cycles), the second window repeats the first window with discontinuity. This discontinuity produces harmonic frequencies that are not present in the original signal (spectral leakage). Figure below shows the spectral plots for both cases using the DFT/FFT directly.

![Figure 5.3 Signal samples and spectra without and with spectral leakage.](image)

The amount of spectral leakage shown in the second plot is due to amplitude discontinuity in time domain. The bigger the discontinuity, the more is the leakage. To reduce the effect of spectral leakage, a window function can be used whose amplitude tapers smoothly and gradually toward zero at both ends. Applying the window function \( w(n) \) to a data sequence \( x(n) \) to obtain a windowed sequence \( x_w(n) \) is better illustrated in the figure below using:

\[
x_w(n) = x(n)w(n), \text{ for } n = 0, 1, \ldots, N - 1
\]
Types of window functions (common functions)

1- The rectangular window (no window function):
\[ w_R(n) = 1 \quad for \quad 0 \leq n \leq N - 1 \]

2- The triangular window:
\[ w_{tri}(n) = 1 - \frac{|2n - N + 1|}{N - 1} \quad for \quad 0 \leq n \leq N - 1 \]

3- The Hamming window:
\[ w_{hm}(n) = 0.54 - 0.46 \cos \left( \frac{2\pi n}{N - 1} \right) \quad for \quad 0 \leq n \leq N - 1 \]

4- The Hanning window:
\[ w_{hn}(n) = 0.5 - 0.5 \cos \left( \frac{2\pi n}{N - 1} \right) \quad for \quad 0 \leq n \leq N - 1 \]
Plots for each window function for a size of 20 samples are shown in Figure below:

![Figure 5.5 Plots of window sequences](image)

**Example**

Consider the sequence $x(0) = 1$, $x(1) = 2$, $x(2) = 3$ and $x(3) = 4$, and given $fs = 100$ Hz, $T = 0.01$ seconds, compute the amplitude spectrum, phase spectrum and power spectrum

a. Using the triangular window function.

b. Using the Hamming window function.

**Solution:**

a) Since $N = 4$, from the triangular window function equation, we have:

$w_{tri}(0) = 0$, $w_{tri}(1) = 0.6667$, $w_{tri}(2) = 0.6667$, and $w_{tri}(3) = 0$.

Now, by applying equation $x_w(n) = x(n)w(n)$, for $n = 0, 1, ..., N - 1$, we have:

$x_w(0) = x(0)w_{tri}(0) = 0$. Similarly $x_w(1) = 1.3334$, $x_w(2) = 2$, and $x_w(3) = 0$

Applying DFT equation to $x_w(n)$ for $K=0, 1, 2$ and $3$, we have:

$X(0) = 3.3334$, $X(1) = -2 - j1.3334$, $X(2) = 0.6666$ and $X(3) = -2 + j1.3334$

$\Delta f = 1 / NT = 25$ Hz

Applying equations of amplitude spectrum, phase spectrum and power spectrum, we get...
\[ A_0 = \frac{1}{4} |X(0)| = 0.8334 \]
\[ P_0 = \frac{1}{4^2} |X(0)|^2 = 0.6954 \]
\[ \varphi_0 = \tan^{-1}\left( \frac{0}{3.3334} \right) = 0^\circ \]

Similarly:

<table>
<thead>
<tr>
<th>K</th>
<th>( A_K )</th>
<th>( \Phi_K \text{ in degree} )</th>
<th>( P_K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6009</td>
<td>-146.31</td>
<td>0.3611</td>
</tr>
<tr>
<td>2</td>
<td>0.1667</td>
<td>0</td>
<td>0.0278</td>
</tr>
<tr>
<td>3</td>
<td>0.6009</td>
<td>146.31</td>
<td>0.3611</td>
</tr>
</tbody>
</table>

b) Since \( N = 4 \), from the Hamming window function, we have:
\[ w_{hm}(0) = 0.08, \ w_{hm}(1) = 0.77, \ w_{hm}(2) = 0.77 \text{ and } w_{hm}(3) = 0.08. \] The windowed sequence is computed using \( x_w(n) = x(n)w(n) \) as:
\[ x_w(0) = x(0)w_{hm}(0) = 0.08, \ x_w(1) = 1.54, \ x_w(2) = 2.31 \text{ and } x_w(3) = 0.32 \]

Applying DFT equation to \( x_w(n) \) for \( K=0, 1, 2 \text{ and } 3 \), we have:
\[ X(0) = 4.25, \ X(1) = -2.23 - j1.22, \ X(2) = 0.53 \text{ and } X(3) = -2.23 + j1.22 \]
\[ \Delta f = \frac{1}{NT} = 25 \text{ Hz} \]

Applying equations of amplitude spectrum, phase spectrum and power spectrum, we get
\[ A_0 = \frac{1}{4} |X(0)| = 1.0625 \]
\[ P_0 = \frac{1}{4^2} |X(0)|^2 = 1.1289 \]
\[ \varphi_0 = \tan^{-1}\left( \frac{0}{4.25} \right) = 0^\circ \]
Similarly:

<table>
<thead>
<tr>
<th>K</th>
<th>$A_K$</th>
<th>$\Phi_K$ in degree</th>
<th>$P_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6355</td>
<td>-151.32</td>
<td>0.4308</td>
</tr>
<tr>
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<td>0.0176</td>
</tr>
<tr>
<td>3</td>
<td>0.6355</td>
<td>151.32</td>
<td>0.4308</td>
</tr>
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</table>