AL FURAT AL AWSAT TECHNICAL UNIVERSITY

NAJAF COLLEGE OF TECHNOLOGY

DEPARTMENT OF TECHNICAL COMMUNICATIONS ENGINEERING

DIGITAL SIGNAL PROCESSING

3rd YEAR

Academic Responsible

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Z - Transform

The z-transform is a very important tool in describing and analyzing digital systems. It also offers the techniques for digital filter design and frequency analysis of digital signals.

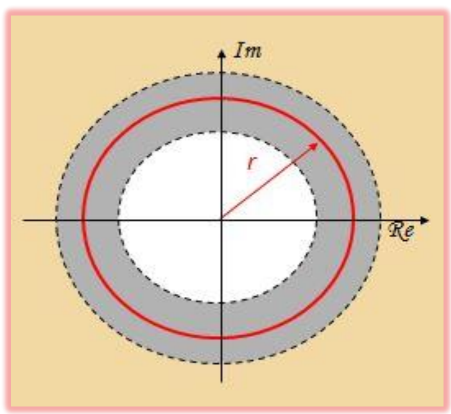
The z-transform of a sequence x(n), designated by X(z) or Z(x(n)), is defined as:

$$\boldsymbol{X(z) = Z\big(x(n)\big) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \cdots + x(-1)z^{1} + x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \cdots}$$

Where, z is the complex variable, n is an integer time index.

Region of Convergence (ROC)

The region of convergence *(ROC)* is the set of points *(z)* in the complex plane, for which the summation is bounded (converges) as shown in the figure below.



$$\left|\sum_{n=-\infty}^{\infty} \boldsymbol{x(n)z^{-n}}\right| < \infty$$

The region of convergence is defined based on the particular sequence x(n) being applied.

Applications of z-transform

This transform is used in many applications of mathematics and signal processing such as:

- Analysis of digital filters.

- Simulating and analysing continuous and discrete systems.

- Used to finding frequency response.

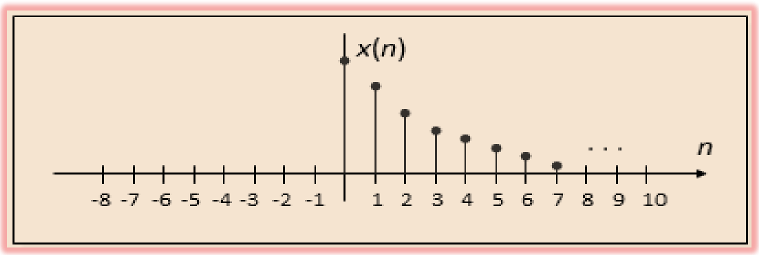
- Analysis of discrete signals.

- Helps in system design and checks the system stability.

Example

Find the Z-transform including region of convergence for the following causal sequence

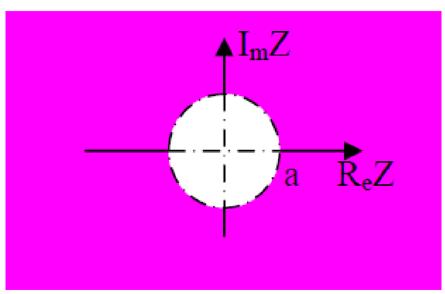
$x(n) = a^n u(n)$



Solution:

$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n).z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$

ROC for the sequence is $|z| > |a|$. The region of convergence (ROC) is outside the unit circle only.



Example

Find Z.T including region of convergence of $x(n) = - a^n u(- n - 1)$.

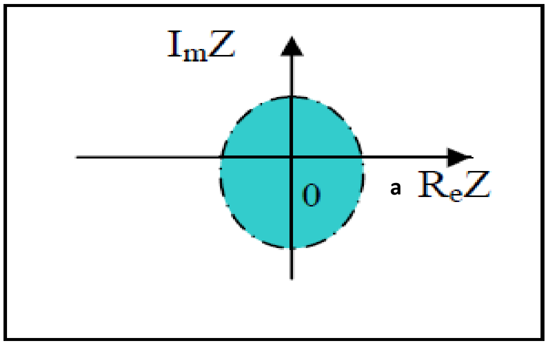
Solution:

The system is non-causal

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u(-n-1).z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{z}{z-a}$$

ROC for the sequence is $|z| < |a|$. The region of convergence (ROC) is inside the unit circle only.



Properties of Z.T

1- Linearity

The z-transform is a linear transformation, which implies

$$Z(ax_1(n) \pm bx_2(n)) = aX_1(z) \pm bX_2(z)$$

Where ***a*** and ***b*** are constants

$ROC = ROC1 \cap ROC2$

2- Shift theorem (without initial conditions)

Given X(z), the z-transform of a sequence x(n), the z-transform of x(n - m), the time-shifted sequence, is given by;

$$z\{x(n-m)\} = z^{-m} X(z)$$

3- Convolution

Given two sequences $x_1(n)$ and $x_2(n)$, their convolution can be determined as follows:

$$x(n) = x_1(n) * x_2(n) \leftrightarrow X(z) = X_1(z)\, X_2(z)$$

Basic Steps:

1. Compute z-Transform of each of the signals to convolve (time domain → z-domain):

$$X_1(z) = Z[x_1(n)] \quad , X_2(z) = Z[x_2(n)]$$

2. Multiply the two z-Transforms (in z-domain): X(z)= X_1 (z) X_2 (z)

3. Find the inverse z-Transform of the product (z-domain → time domain):

$$x(n) = z^{-1}[X(z)]$$

4- Multiplication by exponential

$$Z\{x(n)\} = X(z);$$

$$Z\{e^{\pm an}x(n)\} = X(e^{\mp a}z)$$

5- Initial and final value theorems

If the z-transform of the x(n) is X(z), and if $\lim_{z\to\infty} X(z)$ exists, then the initial value of x(n) (i.e., x(0)) is $x(0) = \lim_{z\to\infty} X(z)$

If the z-transform of the x(n) is X(z), and if $\lim_{n\to\infty} x(n)$ exists, then the value of x(n) as n → ∞ is given by:

$$x(\infty) = \lim_{n\to\infty} x(n) = \lim_{z\to 1}[(z-1).X(z)]$$

6- Multiplication by n (*Differentiation of X(z)*)

$$z\{\, nx(n)\,\} = -z\frac{d}{dz}X(z)$$

Example

Determine the z-transform and the ROC of the signal x(n) = [3(2^n)-4(3^n)] u(n).

Solution:

If we define $x_1(n) = 2^n u(n)$, $x_2(n) = 3^n u(n)$

Then,

$x(n) = [3\,x_1(n) - 4\,x_2(n)]$

$X(z) = 3\ X_1(z) - 4\ X_2(z)$

As we know from z-transform pair $\boldsymbol{a^n u(n)} = \frac{\mathbf{1}}{\mathbf{1-az^{-1}}}$

So, $X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}$ $\qquad$ $ROC\ is\ |z| > 3$