CHAPTER 1

FOURIER SERIES

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Mathematicians of the eighteenth century, including Daniel Bernoulli and Leonard Euler, expressed the problem of the vibratory motion of a stretched string through partial differential equations that had no solutions in terms of ‘‘elementary functions.’’ Their resolution of this difficulty was to introduce infinite series of sine and cosine functions that satisfied the equations. In the early nineteenth century, Joseph Fourier, while studying the problem of heat flow, developed a cohesive theory of such series. Consequently, they were named after him. Fourier series and Fourier integrals are investigated in this and the next chapter. As you explore the ideas, notice the similarities and differences with the chapters on infinite series and improper integrals.

PERIODIC FUNCTIONS

A function $f(t)$ is said to have a period T or to be periodic with period T if for all t, $f(t+T)=f(t)$, where T is a positive constant. The least value of $T>0$ is called the least period or simply the period of $f(t)$.

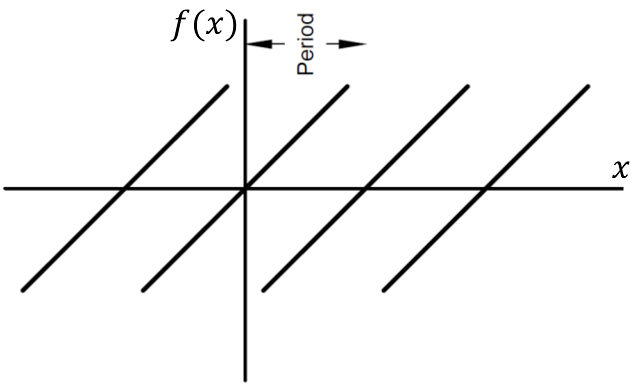
EXAMPLE 1. The function $sin\, t$ has periods $2\pi, 4\pi, 6\pi, \ldots\ldots$ since $sin(t+2\pi)$, $sin(t+4\pi), sin(t+6\pi), \ldots$ all equal $sin\, t$. However, 2π is *the least period* or the period of $sin\, t$.

EXAMPLE 2. The period of $sin\, nx$ or $cos\, nt$ where n is a positive integer, is $2\pi/n$.

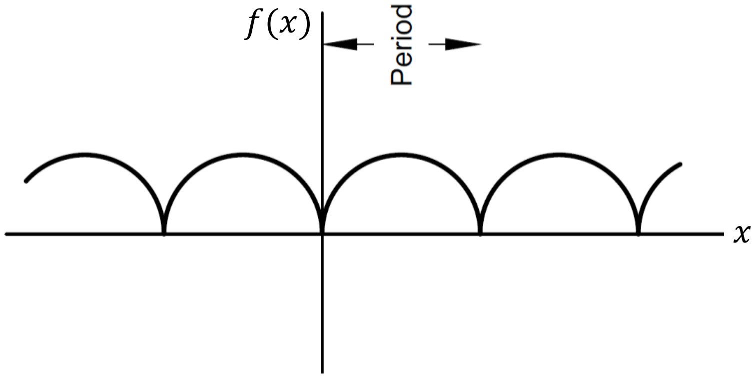
EXAMPLE 3. The period of $tan\, t$ is π.

EXAMPLE 4. A constant has any positive number as period.

Other examples of periodic functions are shown in the graphs of Figures 1(*a*), (*b*), and (*c*) next.



(a)



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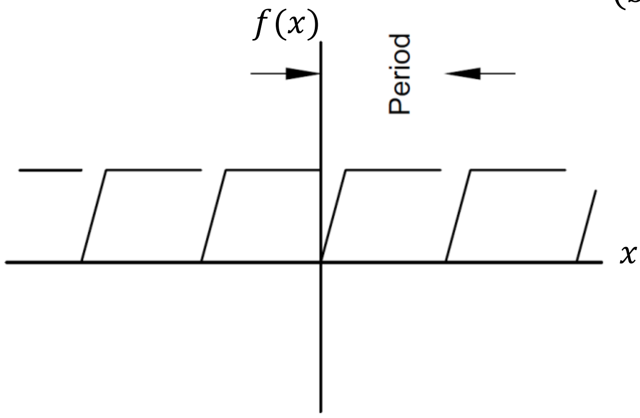


(c)

Example1: find the period of $f(x) = cos\ x$

Solution: $f(x+2\pi) = cos\ (x+2\pi)$

$$= cos\ x\ cos\ 2\pi - sin\ x\ sin\ 2\pi$$

$$= cos\ x$$

$$= f(x)$$

Hence $cos\ x$ is periodic of period 2π

Example 2: find the period of $f(x) = \sin 4x$.

Solution: $f\left(x+\frac{\pi}{2}\right) = sin\ \left(4(x+\frac{\pi}{2})\right)$

$$= sin\ (4x+2\pi)$$

$$= sin\ 4x\ cos\ 2\pi - cos\ 4x\ sin\ 2\pi$$

$$= \sin 4x = f(x)$$

Hence $sin\ nx$ is periodic of period $\frac{\pi}{2}$, observe that 2π is also a period of $sin\ 4x$.

Our purpose is to approximate periodic functions by sine and cosine. We define Fourier series of the periodic function $f(x)$ by:

$$f(x) = a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Fourier coefficients of $f(x)$,given by the Euler formulas:

$$a_o = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

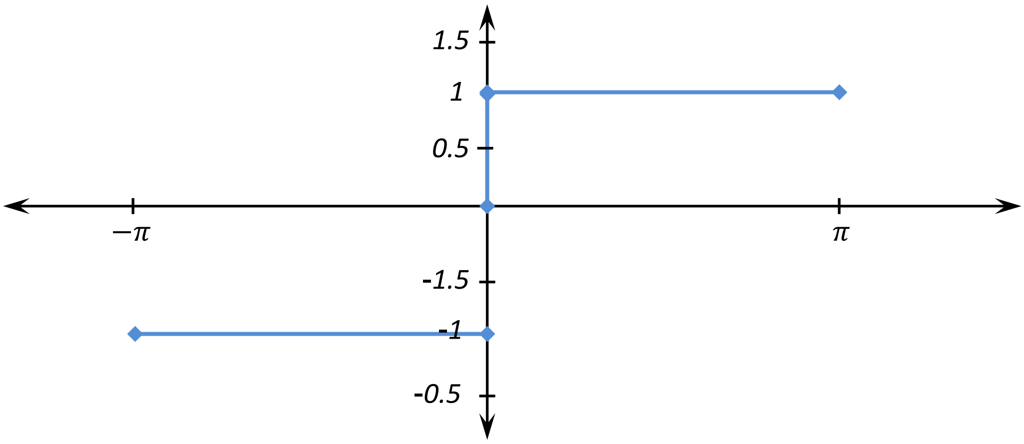
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \;\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \;\, dx$$

Example 3: Find the Fourier series for:

$$f(x) = \begin{cases} 1 & : 0 < x < \pi \\ -1 & : -\pi < x < 0 \end{cases}$$

Solution:



$$a_o = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \;\, dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \; dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} \sin nx \; dx + \int_{0}^{\pi} \sin nx \; dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\cos nx}{n}\bigg|_{-\pi}^{0} - \frac{\cos nx}{n}\bigg|_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left(\frac{1}{n} - \frac{\cos n\pi}{n}\right) - \left(\frac{\cos n\pi}{n} - \frac{1}{n}\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2}{n} - \frac{2\cos n\pi}{n}\right\}$$

$$\cos n\pi = (-1)^n$$

$$= \frac{2}{n\pi}[1 - (-1)^n] = \begin{cases} 0 & : \; n \; is \; even \\ \dfrac{4}{n\pi} & : \; n \; is \; odd \end{cases}$$

Now Fourier series:

$$f(x) = \overbrace{a_o}^{0} + \sum_{n=1}^{\infty}(\overbrace{a_n \cos nx}^{0} + b_n \sin nx)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin(2n-1)x$$

Then we notice:

- $\int_{-\pi}^{\pi} \sin nx \; \sin mx \;\; dx = \begin{cases} 0, \; n \neq m \\ \pi, \; n = m \end{cases}$
- $\int_{-\pi}^{\pi} \sin nx \; \cos mx \;\; dx = 0 \;\; always$
- $\int_{-\pi}^{\pi} \cos nx \; \cos mx \;\; dx = \begin{cases} 0, \; n \neq m \\ \pi, \; n = m \end{cases}$

Functions of Any Period p = 2L

In general:

$$f(x) = a_o + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_o = \frac{1}{2l} \int_{-l}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

Where period, $p = 2l$

Example 4: Find the Fourier series for:

$$f(x) = x^2, \qquad -1 < x < 1$$

Solution:

In this example, $p = 2\ (period = 2)$

In this case when $p = 2L$

Thus in our example $L = 1$

$$a_o = \frac{1}{2} \int_{-1}^{1} x^2\, dx = \frac{1}{3}$$

$$a_n = \frac{1}{1} \int_{-1}^{1} x^2 \cos n\pi x\, dx$$

By Integration by parts:

$$= \frac{2x\cos n\pi x}{n^2\pi^2}\Big|_{-1}^{1}$$

$$= \frac{4(-1)^n}{n^2\pi^2}$$

x^2	$\cos n\pi x$
$2x$	$\frac{\sin n\pi x}{n\pi}$
2	$\frac{-\cos n\pi x}{n^2\pi^2}$
$0 \rightarrow -\int$	$\frac{-\sin n\pi x}{n^3\pi^3}$

$$b_n = \frac{1}{1}\int_{-1}^{1} \overbrace{x^2}^{even} \underbrace{\sin n\pi x}_{odd}\, dx = 0$$

So,

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2}\cos(n\pi x)$$

Fourier Cosine Series

If $f(x)$ is an even function, $(f(-x) = f(x))$, then there will not be any sine terms in the Fourier series for $f(x)$. The Fourier sine coefficient is:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

Since $f(x)$ is an even function and $sin(nx)$ is odd, $f(x)\ sin(nx)$ is odd. b_n is the integral of an odd function from $-\pi$ to π and is thus zero. We can rewrite the cosine coefficients,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$

Example 5: Consider the function defined on $[0, \pi)$ by

$$f(x) = \begin{cases} x & for \quad 0 \le x < \pi/2 \\ \pi - x & for \quad \pi/2 \le x < \pi \end{cases}$$

The Fourier cosine coefficients for this function are:

$$a_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\cos nx \ dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\pi - x)\cos nx\, dx$$

$$= \begin{cases} 0 & n - odd \ \ and\, n - 2,6,10,\dots \\ \dfrac{-8}{n^2\pi} & n - 4,8,12,\dots \end{cases}$$

In *Figure 2* the Fourier cosine series are plotted.

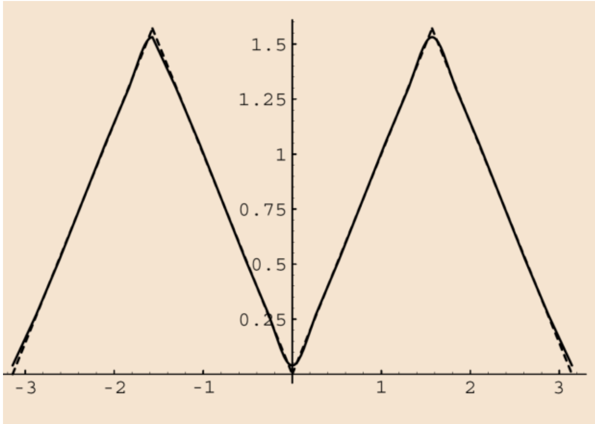


Figure 2: Fourier Cosine Series.

Fourier Sine Series

If $f(x)$ is an odd function, $(f(-x) = -f(x))$, then there will not be any cosine terms in the Fourier series. Since $f(x)\ cos(nx)$ is an odd function, the cosine coefficients will be zero. Since $f(x)\ sin(nx)$ is an even function, we can rewrite the sine coefficients

$$b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \ dx$$

Example 6: Consider the function defined on $[0, \pi)$ by

$$f(x) = \begin{cases} x & for \quad 0 \le x < \pi/2 \\ \pi - x & for \quad \pi/2 \le x < \pi \end{cases}$$

The Fourier sine coefficients for this function are:

$$b_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin nx \ dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin nx\, dx$$

$$= \frac{16}{\pi n^2} \cos\left(\frac{n\pi}{4}\right) \sin^3\left(\frac{n\pi}{4}\right)$$

In *Figure 3* the Fourier sine series are plotted.

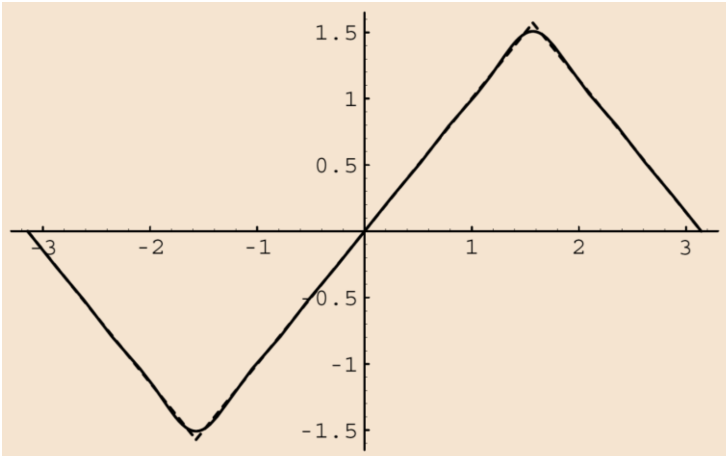


Figure 3: Fourier sine Series.

Example 7: Find the Fourier series for:

$$f(x) = |x| \,, \qquad -\pi < x < \pi$$

Solution:

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| = \frac{2}{\pi}\int_{0}^{\pi} x\, dx = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx \; dx = \frac{2}{\pi}\int_{0}^{\pi} x \cos nx\, dx$$

$$= \frac{2}{\pi}\left(\left[\frac{x \sin nx}{n}\right]_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} dx\right) = \frac{2}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2(-1)^n}{\pi n^2} - \frac{2}{\pi n^2} = \frac{(2(-1)^n - 1)}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin nx \; dx = \frac{1}{\pi}\int_{-\pi}^{0} x \sin nx \; dx + \frac{1}{\pi}\int_{0}^{\pi} x \sin nx \; dx$$

$$= \frac{1}{\pi}\left[\left[\frac{x \cos nx}{n}\right]_{-\pi}^{0} - \int_{-\pi}^{0} \frac{\cos nx}{n} dx\right] + \frac{1}{\pi}\left[\left[\frac{-x \cos nx}{n}\right]_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi(-1)^n}{n} - \left[\frac{\sin nx}{n^2}\right]_{-\pi}^{0}\right] + \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \left[\frac{\sin nx}{n^2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\frac{\pi(-1)^n}{n} + \frac{1}{\pi}\frac{-\pi(-1)^n}{n} = 0$$

So on the interval $[-\pi, \pi]$, our function is:

$$|x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{2((-1)^n - 1)}{\pi n^2}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$$