Lecture- 08

THE BERNOULLI EQUATION

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Learning Objectives

- After completing this Lecture, you should be able to:

1. Define The Bernoulli equation
2. Derivation of the Bernoulli Equation
3. Understanding the Limitations on the Use of the Bernoulli Equation.
4. Applications of the Bernoulli Equation

Outline

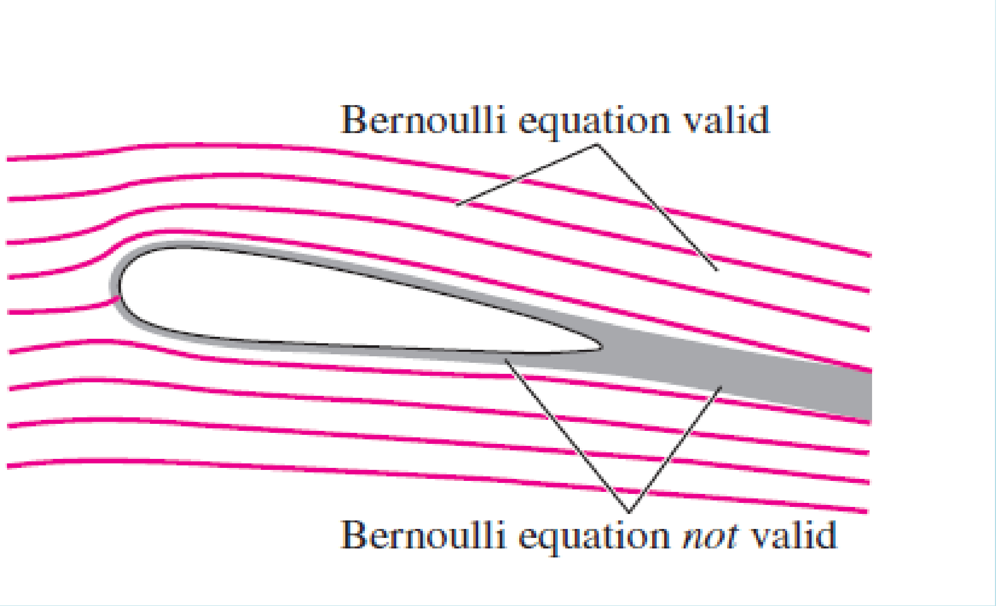
- Overview
- THE BERNOULLI EQUATION
- Limitations on the Use of the Bernoulli Equation.
- Static, Stagnation, Dynamic, and Total Pressure: Bernoulli Equation
- Stagnation Point: Bernoulli Equation
- Pitot-Static Tube: Speed of Flow
- Pitot-Static Tube: Design
- Uses of Bernoulli Equation: Free Jets
- Uses of Bernoulli Equation: Flow Rate Measurement

The *Bernoulli equation* is concerned with the conservation of kinetic, potential, and flow energies of a fluid stream and their conversion to each other in regions of flow where net viscous forces are negligible and where other restrictive conditions apply. The *energy equation* is a statement of the conservation of energy principle.

THE BERNOULLI EQUATION

- The **Bernoulli equation** is *an approximate relation between pressure,velocity, and elevation,* and is valid in *regions of steady, incompressible flow where net frictional forces are negligible* (as shown in the Figure below). Despite its simplicity, it has proven to be a very powerful tool in fluid mechanics.

The *Bernoulli equation* is an *approximate* equation that is valid only in *in viscid regions of flow* where net viscous forces are negligibly small compared to inertial, gravitational, or pressure forces. Such regions occur outside of *boundary layers* and *wakes*.



Derivation of the Bernoulli Equation

the Bernoulli Equation is derived from the mechanical energy equation

$$\Delta e_{\text{mech}} = \frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) \qquad \text{(kJ/kg)}$$

since the we are dealing with steady flow system with out the effect of the mechanical work and the friction on the system the first terms become zero.

$$\textit{Steady, incompressible flow:} \quad \frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)}$$

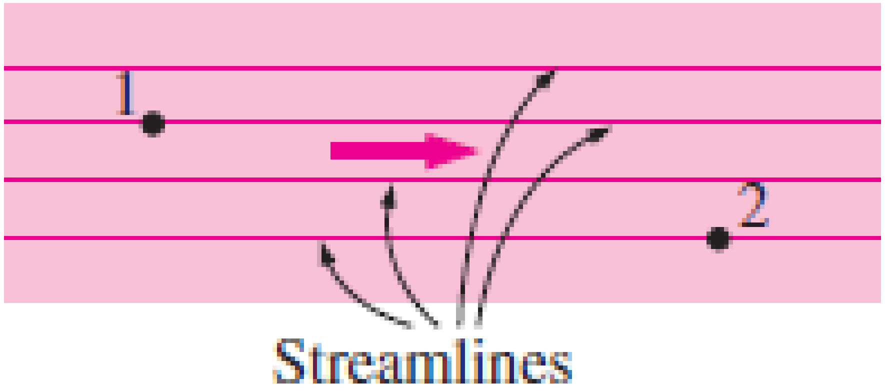
This is the famous **Bernoulli equation,** which is commonly used in fluid mechanics for steady, incompressible flow along a streamline in inviscid regions of flow.

The Bernoulli equation can also be written between any two points on the same streamline as

Steady, incompressible flow: $$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Limitations on the Use of the Bernoulli Equation

- **Steady flow** The first limitation on the Bernoulli equation is that it is applicable to *steady flow*.
- **Frictionless flow** Every flow involves some friction, no matter how small, and *frictional effects* may or may not be negligible.
- **No shaft work** The Bernoulli equation was derived from a force balance on a particle moving along a streamline.
- **Incompressible flow** One of the assumptions used in the derivation of the Bernoulli equation is that $\rho =$ constant and thus the flow is incompressible.
- **No heat transfer** The density of a gas is inversely proportional to temperature, and thus the Bernoulli equation should not be used for flow sections that involve significant temperature change such as heating or cooling sections.
- Strictly speaking, the Bernoulli equation $P/\rho + V^2/2 + gz = C$ is applicable along a streamline, and the value of the constant *C*, in general, is different for different streamlines. But when a region of the flow is *irrotational*, and thus there is no *vorticity* in the flow field, the value of the constant *C* remains the same for all streamlines, and, therefore, the Bernoulli equation becomes applicable *across* streamlines as well.



$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

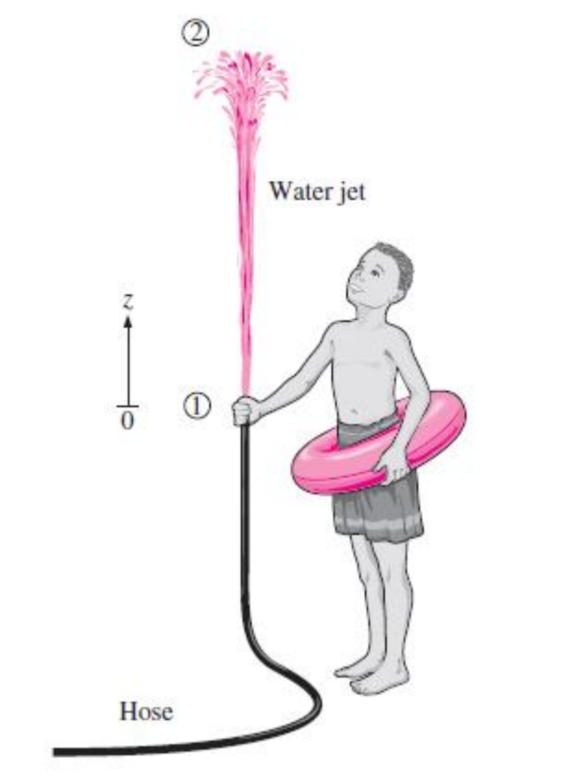
FIGURE 5–32

When the flow is irrotational, the Bernoulli equation becomes applicable between any two points along the flow (not just on the same streamline).

EXAMPLE 2–2 **Spraying Water into the Air**

Water is flowing from a hose attached to a water main at 400 kPa gage (Fig. below). A child places his thumb to cover most of the hose outlet, causing a thin jet of high-speed water to emerge. If the hose is held upward, what is the maximum height that the jet could achieve?

This problem involves the conversion of flow, kinetic, and potential energies to each other without involving any pumps, turbines, and wasteful components with large frictional losses, and thus it is suitable for the use of the Bernoulli equation. The water height will be maximum under the stated assumptions. The velocity inside the hose is relatively low ($V1 = 0$) and we take the hose outlet as the reference level ($z1 = 0$). At the top of the water trajectory $V2 = 0$, and atmospheric pressure pertains. Then the Bernoulli equation simplifies to



$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g}\nearrow^0 + z_1\nearrow^0 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g}\nearrow^0 + z_2 \quad\rightarrow\quad \frac{P_1}{\rho g} = \frac{P_{atm}}{\rho g} + z_2$$

Solving for z_2 and substituting,

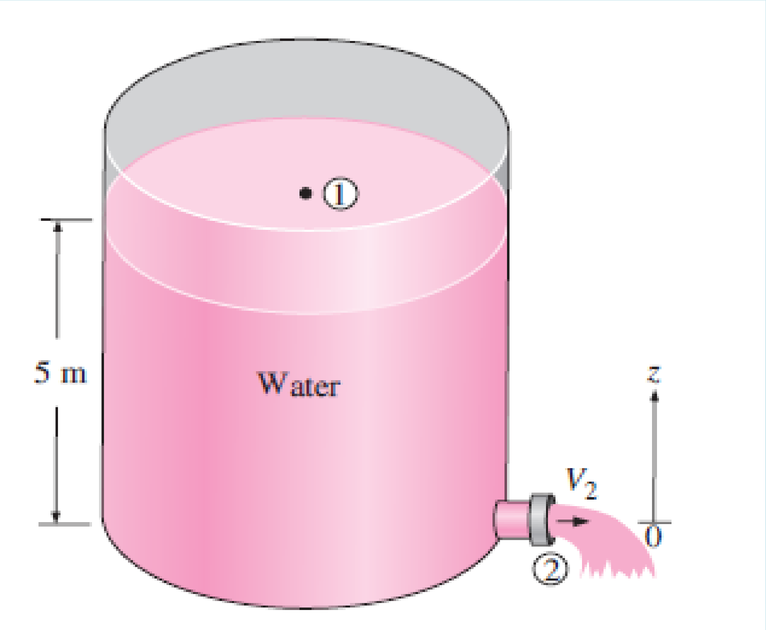
$$z_2 = \frac{P_1 - P_{atm}}{\rho g} = \frac{P_{1,\,gage}}{\rho g} = \frac{400 \text{ kPa}}{(1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)}\left(\frac{1000 \text{ N/m}^2}{1 \text{ kPa}}\right)\left(\frac{1 \text{ kg}\cdot\text{m/s}^2}{1 \text{ N}}\right)$$

$$= 40.8 \text{ m}$$

EXAMPLE 2-3 **Water Discharge from a Large Tank**

A large tank open to the atmosphere is filled with water to a height of 5 m from the outlet tap (Fig. below). A tap near the bottom of the tank is now opened, and water flows out from the smooth and rounded outlet. Determine the water velocity at the outlet.

This problem involves the conversion of flow, kinetic, and potential energies to each other without involving any pumps, turbines, and wasteful components with large frictional losses, and thus it is suitable for the use of the Bernoulli equation. We take point 1 to be at the free surface of water so that $P_{1=}\ P_{atm}$ (open to the atmosphere), $V_1 = 0$ (the tank is large relative to the outlet), and $z_{1=}\ 5$ m and $z_2 = 0$ (we take the reference level at the center of the outlet). Also, $P_2 = P_{atm}$ (water discharges into the atmosphere).



Then the Bernoulli equation simplifies to

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g}^{\nearrow 0} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2^{\nearrow 0} \quad \rightarrow \quad z_1 = \frac{V_2^2}{2g}$$

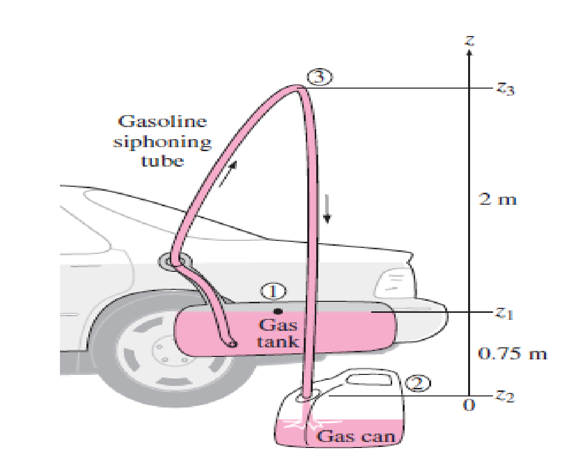
Solving for V_2 and substituting

$$V_2 = \sqrt{2gz_1} = \sqrt{2(9.81\ \text{m/s}^2)(5\ \text{m})} = \mathbf{9.9\ m/s}$$

The relation $V = \sqrt{2gz}$ is called the **Toricelli equation.**

_EXAMPLE 2–4_Siphoning Out Gasoline from a Fuel Tank

During a trip to the beach (P_{atm} = 1 atm =101.3 kPa), a car runs out of gasoline, and it becomes necessary to siphon gas out of the car of a Good Samaritan (Fig. below). The siphon is a small-diameter hose, and to start the siphon it is necessary to insert one siphon end in the full gas tank, fill the hose with gasoline via suction, and then place the other end in a gas can below the level of the gas tank. The difference in pressure between point 1 (at the free surface of the gasoline in the tank) and point 2 (at the outlet of the tube) causes the liquid to flow from the higher to the lower elevation. Point 2 is located 0.75 m below point 1 in this case, and point 3 is located 2 m above point 1. The siphon diameter is 4 mm, and frictional losses in the siphon are to be disregarded. Determine (*a*) the minimum time to withdraw 4 L of gasoline from the tank to the can and (*b*) the pressure at point 3. The density of gasoline is 750 kg/m^3.



Analysis (*a*) We take point 1 to be at the free surface of gasoline in the tank so that $P_1 = P_{atm}$ (open to the atmosphere), $V_1 = 0$ (the tank is large relative to the tube diameter), and $z_2 = 0$ (point 2 is taken as the reference level). Also, $P_2 = P_{atm}$ (gasoline discharges into the atmosphere). Then the Bernoulli equation simplifies to

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g}^{\nearrow 0} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2^{\nearrow 0} \quad \rightarrow \quad z_1 = \frac{V_2^2}{2g}$$

Solving for V_2 and substituting,

$$V_2 = \sqrt{2gz_1} = \sqrt{2(9.81 \text{ m/s}^2)(0.75 \text{ m})} = 3.84 \text{ m/s}$$

The cross-sectional area of the tube and the flow rate of gasoline are

$$A = \pi D^2/4 = \pi(5 \times 10^{-3} \text{ m})^2/4 = 1.96 \times 10^{-5} \text{ m}^2$$

$$\dot{V} = V_2 A = (3.84 \text{ m/s})(1.96 \times 10^{-5} \text{ m}^2) = 7.53 \times 10^{-5} \text{ m}^3\text{/s} = 0.0753 \text{ L/s}$$

Then the time needed to siphon 4 L of gasoline becomes

$$\Delta t = \frac{V}{\dot{V}} = \frac{4 \text{ L}}{0.0753 \text{ L/s}} = \mathbf{53.1 \text{ s}}$$

(b) The pressure at point 3 can be determined by writing the Bernoulli equation between points 2 and 3. Noting that $V_2 = V_3$ (conservation of mass), $z_2 = 0$, and $P_2 = P_{atm}$,

$$\frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \overset{0}{\nearrow} = \frac{P_3}{\rho g} + \frac{V_3^2}{2g} + z_3 \quad \rightarrow \quad \frac{P_{atm}}{\rho g} = \frac{P_3}{\rho g} + z_3$$

Solving for P_3 and substituting,

$$P_3 = P_{atm} - \rho g z_3$$

$$= 101.3 \text{ kPa} - (750 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(2.75 \text{ m})\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right)\left(\frac{1 \text{ kPa}}{1000 \text{ N/m}^2}\right)$$

$$= \mathbf{81.1 \text{ kPa}}$$

Static, Stagnation, Dynamic, and Total Pressure: Bernoulli Equation

$$p + \tfrac{1}{2}\rho V^2 + \gamma z = \text{constant along streamline}$$

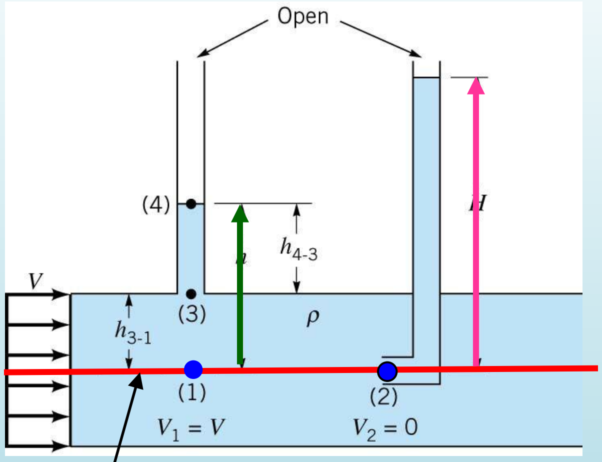
- p → **Static Pressure**
- $\tfrac{1}{2}\rho V^2$ → **Dynamic Pressure**
- γz → **Hydrostatic Pressure**

Static Pressure: moves along the fluid "static" to the motion.

$$p_1 = \gamma h$$

Dynamic Pressure: due to the mean flow going to forced stagnation.

Hydrostatic Pressure: potential energy due to elevation changes.



Follow a Streamline from point 1 to 2

Following a streamline:

$$p_2 + \frac{1}{2}\rho V_2^{\,2} + \gamma z_2 = p_1 + \frac{1}{2}\rho V_1^{\,2} + \gamma z_1$$

$\rho V_2^{\,2}$ → 0; γz_2 → 0, no elevation; γz_1 → 0, no elevation

$$p_2 = p_1 + \frac{1}{2}\rho V_1^{\,2}$$

Note: $p_2 = \gamma H$ **"Total Pressure = Dynamic Pressure + Static Pressure"**

H > h

$$V_1 = \sqrt{2g\left(H - h\right)}$$

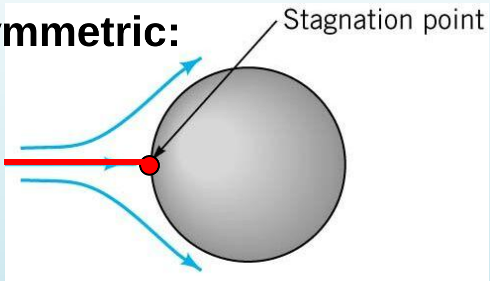
In this way we obtain a measurement of the centerline flow with piezometer tube.

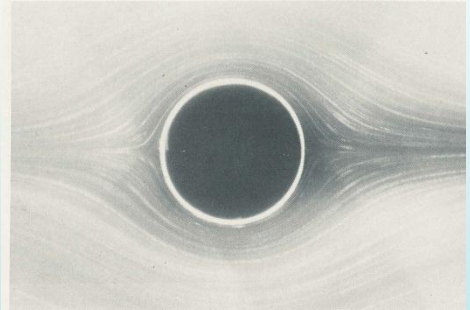
Stagnation Point: Bernoulli Equation

Stagnation point: the point on a stationary body in every flow where V= 0

Stagnation Streamline: The streamline that terminates at the stagnation point.

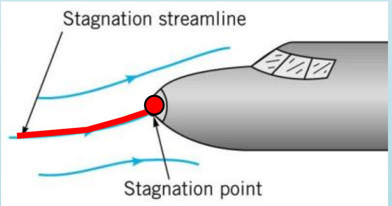
Symmetric:





Stagnation Flow I:

Axisymmetric:



Stagnation Flow II:

If there are no elevation effects, the stagnation pressure is largest pressure obtainable along a streamline: all kinetic energy goes into a pressure rise:

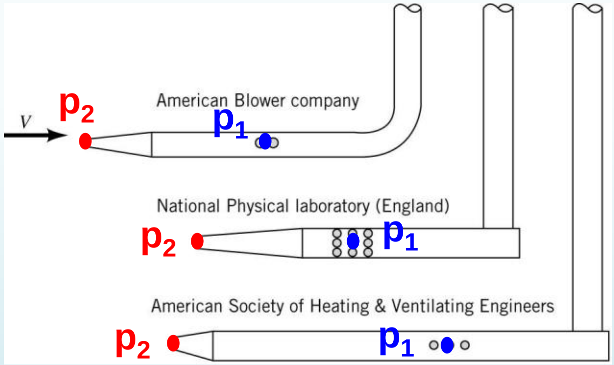
$$p + \frac{\rho V^2}{2}$$

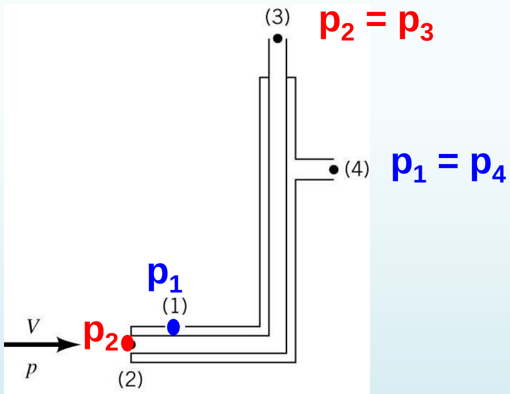
Total Pressure with Elevation:

$$p + \frac{1}{2}\rho V^2 + \gamma z = p_T = constant \;\; on \;\; a \;\; streamline$$

Pitot-Static **Tube: Speed of Flow**

H. De Pitot
(1675-1771)





Stagnation Pressure occurs at tip of the Pitot-static tube:

$$p_2 = p + \frac{1}{2}\rho V^2 = p_3$$

Static Pressure occurs along the static ports on the side of the tube:

$p_1 = p = p_4$ (if the elevation differences are negligible, i.e. air)

Now, substitute static pressure in the stagnation pressure equation:

$$p_3 = p_4 + \frac{1}{2}\rho V^2 \Rightarrow p_3 - p_4 = \frac{1}{2}\rho V^2$$

Now solve for V:

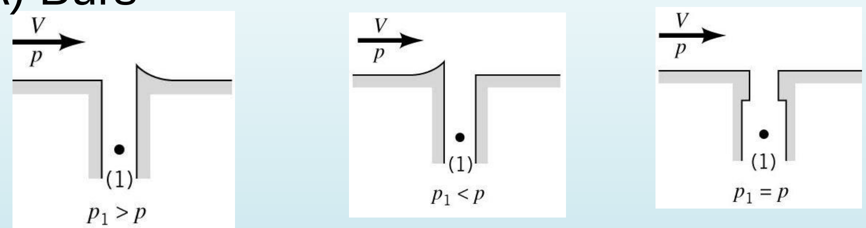
$$V = \sqrt{2(p_3 - p_4)/\rho}$$

Air Speed:

Pitot-Static Tube: Design

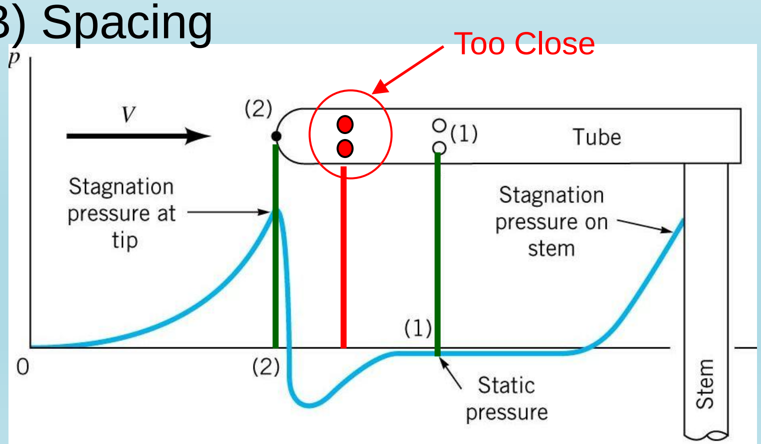
- Pitot-static probes are relatively simple and inexpensive
- Depends on the ability to measure static and stagnation pressure
- The pressure values must be obtained very accurately

Sources of Error in Design in the Static Port:
A) Burs



Error: Stagnates **Error: Accelerates** **OK**

Sources of Error in Design in the Static Port:
B) Spacing

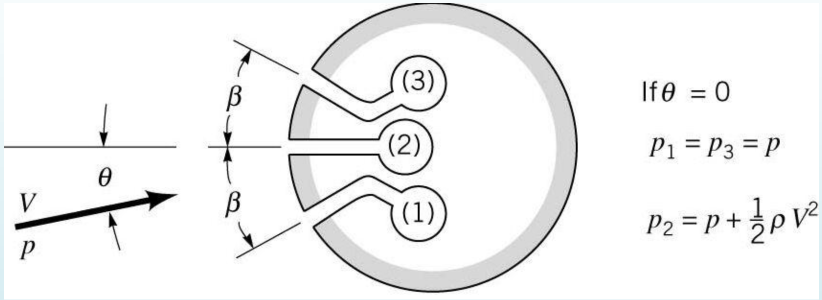


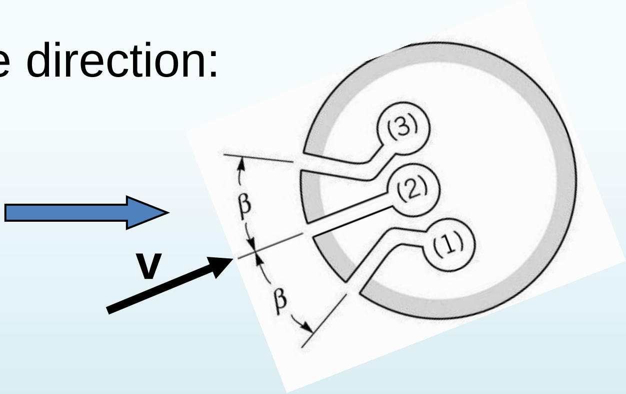
Sources of Error in Design in the Static Port: C) Alignment in Flow

Yaw Angle of 12 to 20° result in less than 1% error.

Pitot-Static Tube: Direction of Flow

A Pitot-static Probe to determine direction:





Rotation of the cylinder until p_1 and p_3 are the same indicating the hole in the center is pointing directly upstream.

P_2 is the stagnation pressure and p_1 and p_3 measure the static pressure.

β is at the angle to p_1 and p_3 and is at 29.5°.

The equation with this type of pitot-static probe is the follwing:

$$V = \sqrt{2(p_2 - p_1)/\rho}$$

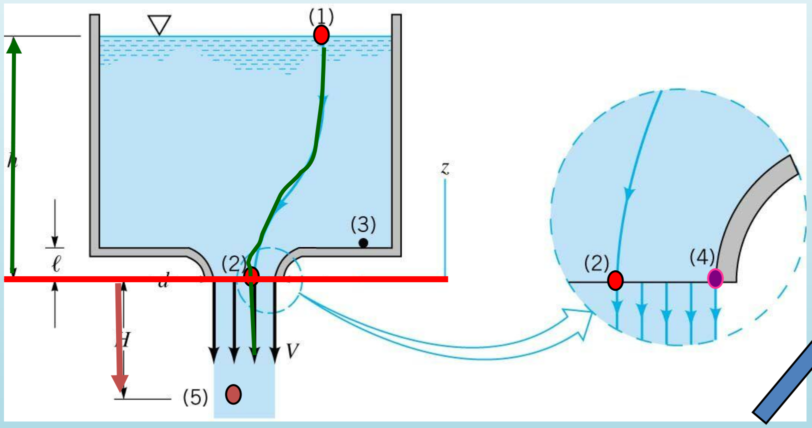
Uses of Bernoulli Equation: Free Jets

New form for along a streamline between any two points:

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

If we know 5 of the 6 variable we can solve for the last one.

Free Jets: Case 1



$$\gamma h = \frac{1}{2}\rho V^2$$

Torricelli's Equation (1643):

$$V = \sqrt{2\frac{\gamma h}{\rho}} = \sqrt{2gh}$$

Note: $p_2 = p_4$ by normal to the streamline since the streamlines are straight.

Following the streamline between (1) and (2):

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

0 gage **0** **h** **0 gage** **V** **0**

As the jet falls:

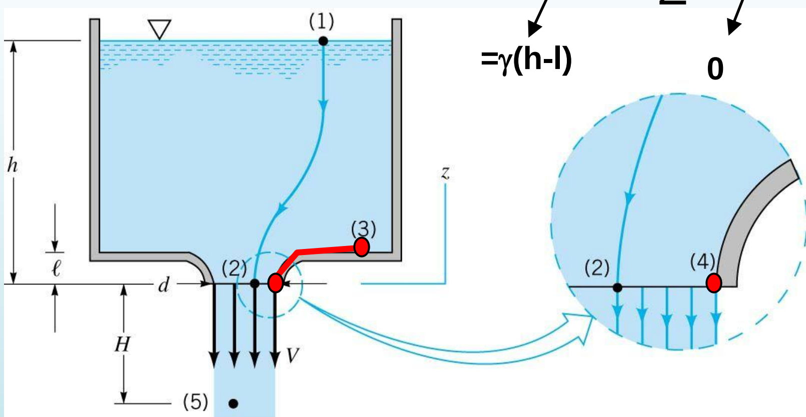
$$V = \sqrt{2g(h + H)}$$

Uses of Bernoulli Equation: Free Jets

Free Jets: Case 2

$$p_3 + \frac{1}{2}\rho V_3^{\,2} + \gamma z_3 = p_4 + \frac{1}{2}\rho V_4^{\,2} + \gamma z_4$$

=γ(h-l) 0 l 0 gage V 0



Then,

$$V = \sqrt{2gh}$$

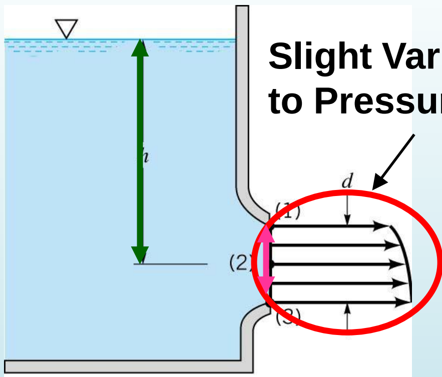
Physical Interpretation:

All the particles potential energy is converted to kinetic energy assuming no viscous dissipation.

The potential head is converted to the velocity head.

Uses of Bernoulli Equation: Free Jets

Free Jets: Case 3 "Horizontal Nozzle: Smooth Corners"



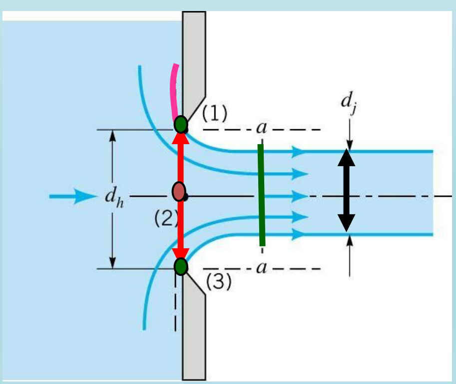
Slight Variation in Velocity due to Pressure Across Outlet

Torricelli Flow:

However, we calculate the average velocity at h, if h >> d:

$$V = \sqrt{2\frac{\gamma h}{\rho}} = \sqrt{2gh}$$

Free Jets: Case 4 "Horizontal Nozzle: Sharp-Edge Corners"



vena contracta: The diameter of the jet d_j is less than that of the hole d_h due to the inability of the fluid to turn the 90° corner.

The pressure at (1) and (3) is zero, and the pressure varies across the hole since the streamlines are curved.

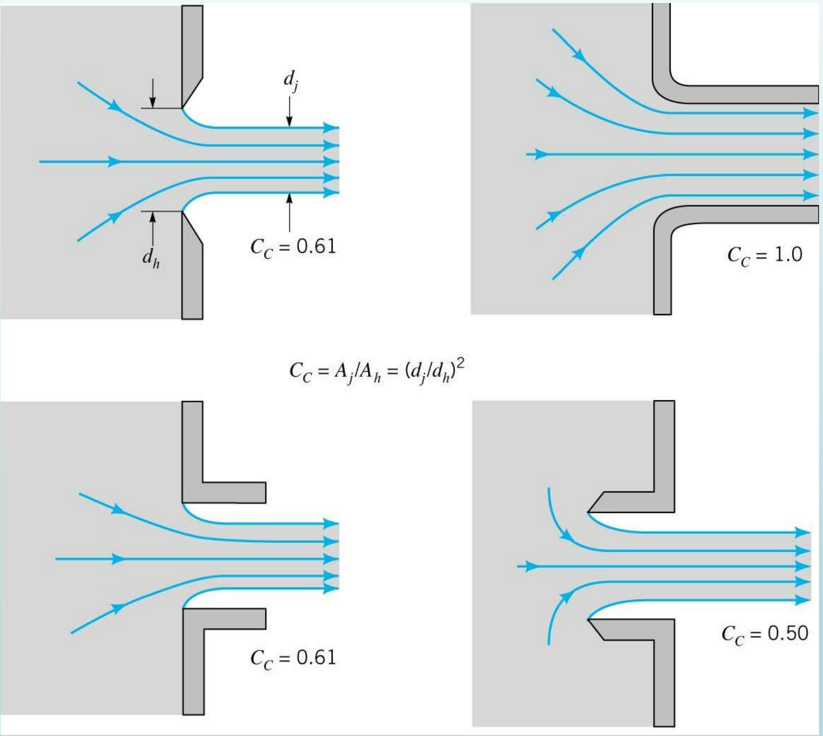
The pressure at the center of the outlet is the greatest.

However, in the jet the pressure at a-a is uniform, we can us Torrecelli's equation if d_j << h.

Uses of Bernoulli Equation: Free Jets

Free Jets: Case 4 “Horizontal Nozzle: Sharp-Edge Corners”

Vena-Contracta Effect and Coefficients for Geometries



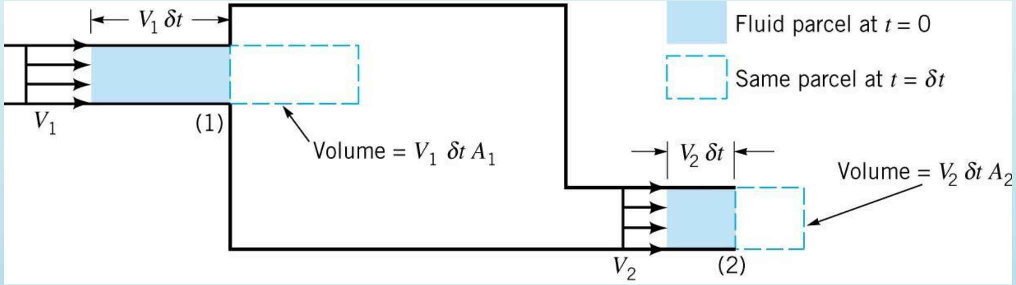
$$C_C = A_j/A_h = (d_j/d_h)^2$$

Uses of the Bernoulli Equation: Confined Flows

There are some flow where we can-not know the pressure a-priori because the system is confined, i.e. inside pipes and nozzles with changing diameters.

In order to address these flows, we consider both conservation of mass (continuity equation) and Bernoulli's equation.

Consider flow in and out of a Tank:



The mass flow rate in must equal the mass flow rate out for a steady state flow:

$\dot{m}$ (slugs/s or kg/s) $\quad \dot{m} = \rho Q$ and $Q = VA$

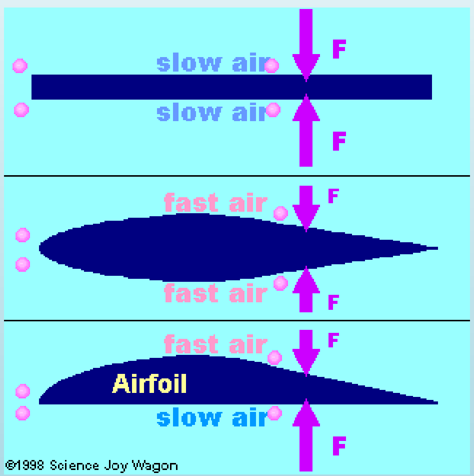
$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

With constant density, $A_1 V_1 = A_2 V_2$, or $Q_1 = Q_2$

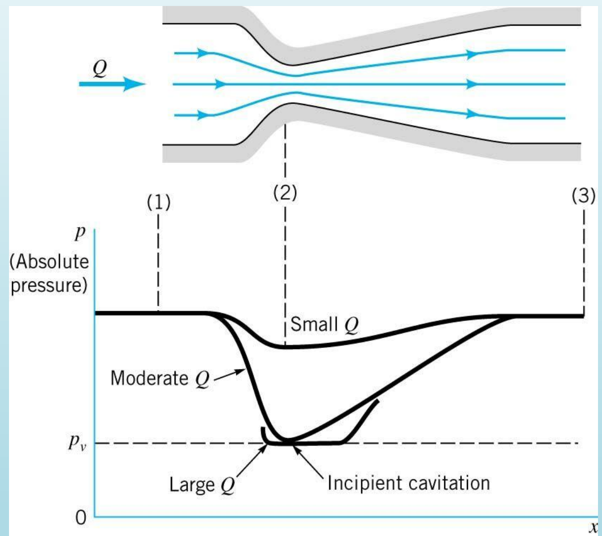
Uses of the Bernoulli Equation: Final Comments

In general, an increase in velocity results in a decrease in pressure.

Airplane Wings:



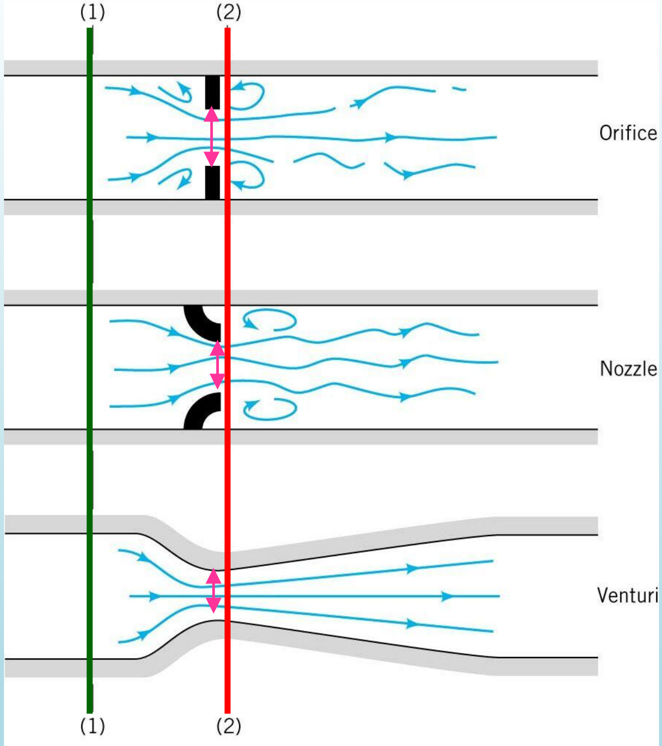
Flow in a Pipe:



Venturi Flow:

Uses of Bernoulli Equation: Flow Rate Measurement

Flowrate Measurements in Pipes using Restriction:



Horizontal Flow:

An increase in velocity results in a decrease in pressure.

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$$

Assuming conservation of mass:

$$Q = A_1 V_1 = A_2 V_2$$

Substituting we obtain:

$$Q = A_2 \sqrt{\frac{2(p_1 - p_2)}{\rho[1 - (A_2/A_1)^2]}}$$

So, if we measure the pressure difference between (1) and (2) we have the flow rate.