Control Systems (CS)

Lecture-14
Routh-Herwitz Stability Criterion (Part A)

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Routh-Hurwitz Stability Criterion

- It is a method for determining continuous system stability.

- The Routh-Hurwitz criterion states that “the number of roots of the characteristic equation with positive real parts is equal to the number of changes in sign of the first column of the Routh array”.

Routh-Hurwitz Stability Criterion

- This method yields stability information without the need to solve for the closed-loop system poles.
- Using this method, we can tell how many closed-loop system poles are in the left half-plane, in the right half-plane, and on the jw-axis. (Notice that we say how many, not where.)

- The method requires two steps:

1. Generate a data table called a Routh table.
2. interpret the Routh table to tell how many closed-loop system poles are in the LHP, the RHP, and on *the jw-axis.*

Routh-Hurwitz Stability Criterion

- The characteristic equation of the n^th order continuous system can be written as:

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

- The stability criterion is applied using a Routh table which is defined as;

$$\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
\cdot & b_1 & b_2 & b_3 & \cdots \\
\cdot & c_1 & c_2 & c_3 & \cdots \\
\cdot & \cdots & \cdots & \cdots & \cdots
\end{array}$$

- Where $a_n, a_{n-1}, \ldots, a_0$ re coefficients of the characteristic equation.

$$b_1 \equiv \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \qquad b_2 \equiv \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 \equiv \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1} \qquad c_2 \equiv \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$

Generating a Basic Routh Table

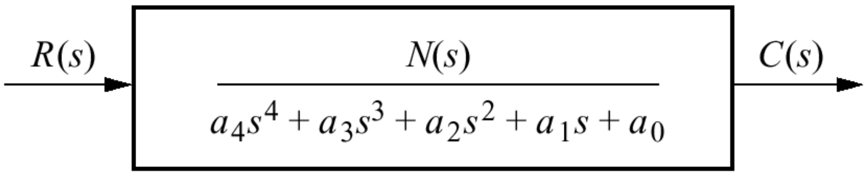
- First label the rows with powers of s from highest power of s down to lowest power of s in a vertical column.
- Next form the first row of the Routh table, using the coefficients of the denominator of the closed-loop transfer function (characteristic equation).
- Start with the coefficient of the highest power and skip every other power of s.
- Now form the second row with the coefficients of the denominator skipped in the previous step.
- The table is continued horizontally and vertically until zeros are obtained.
- For convenience, any row can be multiplied or divide by a positive constant before the next row is computed without changing the values of the rows below and disturbing the properties of the Routh table.

Routh’s Stability Condition

- If the closed-loop transfer function has all poles in the left half of the s-plane, the system is stable. Thus, a system is stable if there are no sign changes in the first column of the Routh table.

- The Routh-Hurwitz criterion declares that the number of roots of the polynomial that are lies in the right half-plane is equal to the number of sign changes in the first column. Hence the system is unstable if the poles lies on the right hand side of the s-plane.

Example: Generating a basic Routh Table.



- Only the first 2 rows of the array are obtained from the characteristic eq. the remaining are calculated as follows;

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Four Special Cases or Configurations in the First Column Array of the Routh's Table:

1. **Case-I:** No element in the first column is zero.
2. **Case-II:** A zero in the first column but some other elements of the row containing the zero in the first column are nonzero.
3. **Case-III:** A zero in the first column and the other elements of the row containing the zero are also zero.
4. **Case-IV:** As in the third case but with repeated roots on the $j\omega$-axis.

Case-I: No element in the first column is zero.

Second-Order System.

The characteristic polynomial of a second order system is given below

$$q(s) = a_2 s^2 + a_1 s + a_0$$

The Routh array is written as

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & b_1 & 0 \end{array}$$

Where

$$b_1 = \frac{a_1 a_0 - (0) a_2}{a_1} = \frac{-1}{a_1} \begin{vmatrix} a_2 & a_0 \\ a_1 & 0 \end{vmatrix} = a_0$$

The requirement for a stable second order system is simply that all the coefficient be positive or all the coefficient s be negative.

Third-Order System.

The characteristic polynomial of a third order system is given below

$$q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

The Routh array is

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & b_1 & 0 \\ s^0 & c_1 & 0 \end{array}$$

Where

$$b_1 = \frac{a_2 a_1 - a_0 a_3}{a_2} \quad \text{and} \quad c_1 = \frac{b_1 a_0}{b_1} = a_0$$

- The requirement for a stable third order system is that the coefficients be positive and $a_2 a_1 > a_0 a_3$.
- The condition when $a_2 a_1 = a_0 a_3$sults in a marginally stability case (recognized as Case-3 because there is a zero in the first column) and one pair of roots lies on the imaginary axis in the s-plane.

Example-1: Find the stability of the continues system having the characteristic equation of

$$s^3 + 6s^2 + 12s + 8 = 0$$

The Routh table of the given system is computed and shown is the table below;

$$\begin{array}{c|ccc}
s^3 & 1 & 12 & 0 \\
s^2 & 6 & 8 & 0 \\
s^1 & \frac{64}{6} & 0 & \\
s^0 & 8 & &
\end{array}$$

- Since there is no changes of the sign in the first column of the Routh table, it means that all the roots of the characteristic equation have negative real parts and hence this system is stable.

Example-2: Find the stability of the continues system having the characteristic polynomial of a third order system is given below

$$s^3 + s^2 + 2s + 24$$

- The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 24 \\ s^1 & -22 & 0 \\ s^0 & 24 & 0 \end{array}$$

- Because TWO changes in sign appear in the first column, we find that two roots of the characteristic equation lie in the right hand side of the s-plane. Hence the system is unstable.

Example-3: Determine a rang of values of a system parameter K for which the system is stable.

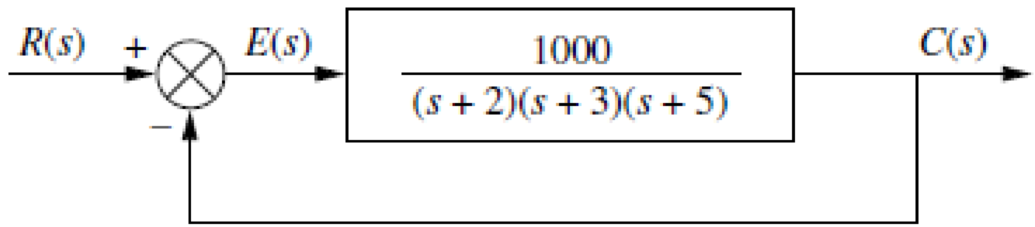
$$s^3 + 3s^2 + 3s + 1 + K = 0$$

- The Routh table of the given system is computed and shown is the table below;

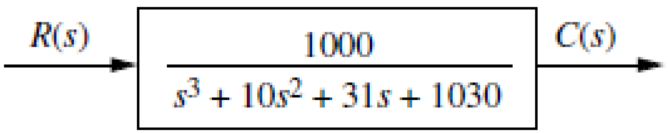
$$\begin{array}{c|ccc} s^3 & 1 & 3 & 0 \\ s^2 & 3 & 1+K & 0 \\ s^1 & \dfrac{8-K}{3} & 0 & \\ s^0 & 1+K & & \end{array}$$

- For system stability, it is necessary that the conditions $8 - k > 0$, and $1 + k > 0$, must be satisfied. Hence the rang of values of a system parameter k must be lies between -1 and 8 (i.e., $-1 < k < 8$).

Example-4: Find the stability of the system shown below using Routh criterion.



The close loop transfer function is shown in the figure



The Routh table of the system is shown in the table

s^3	1	31	0
s^2	$\cancel{10}\quad 1$	$\cancel{1030}\quad 103$	0
s^1	$\dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

Because TWO changes in sign appear in the first column, we find that two roots of the characteristic equation lie in the right hand side of the s-plane. Hence the system is unstable.