Control Systems (CS)

Lecture-15
Routh-Herwitz Stability Criterion (Part B)

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Routh-Hurwitz Stability Criterion

- It is a method for determining continuous system stability.

- The Routh-Hurwitz criterion states that “the number of roots of the characteristic equation with positive real parts is equal to the number of changes in sign of the first column of the Routh array”.

Example-5: Find the stability of the system shown below using Routh criterion.

$$2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

- The Routh table of the system is

s^4	2	3	10
s^3	1	5	0
s^2	$\dfrac{(1)(3)-(2)(5)}{1} = -7$	10	0
s^1	$\dfrac{(-7)(5)-(1)(10)}{-7} = 6.43$	0	0
s^0	10	0	0

- System is unstable because there are two sign changes in the first column of the Routh's table. Hence the equation has two roots on the right half of the s-plane.

Case-II: A Zero Only in the First Column

There are TWO methods in case-II.

1. Stability via Epsilon Method.
2. Stability via Reverse Coefficients (Phillips, 1991).

Case-II: Stability via Epsilon Method

- If the first element of a row is zero, division by zero would be required to form the next row.
- To avoid this phenomenon, an ***epsilon, ε,*** (a small positive number) is assigned to replace the zero in the first column.
- The value ε is then allowed to approach zero from either the positive or the negative side, after which the signs of the entries in the first column can be determined.

Case-II: Stability via Epsilon Method

Example-6: Determine the stability of the system having a characteristic equation given below;

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

The Routh array is shown in the table;

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & \epsilon & 6 & 0 \\ s^2 & c_1 & 10 & 0 \\ s^1 & d_1 & 0 & 0 \\ s^0 & 10 & 0 & 0 \end{array}$$

Where

$$c_1 = \frac{4\epsilon - 12}{\epsilon} = \frac{-12}{\epsilon} \quad \text{and} \quad d_1 = \frac{6c_1 - 10\epsilon}{c_1} \rightarrow 6.$$

There are TWO sign changes due to the large negative number in the first column, $c_1 = -12/\epsilon.$
Therefore the system is unstable, and two roots of the equation lie in the right half of the s-plane.

Example-7: Determine the range of parameter *K* for which the system is unstable.

$$q(s) = s^4 + s^3 + s^2 + s + K$$

The Routh array of the above characteristic equation is shown below;

$$\begin{array}{c|ccc}
s^4 & 1 & 1 & K \\
s^3 & 1 & 1 & 0 \\
s^2 & \epsilon & K & 0, \\
s^1 & c_1 & 0 & 0 \\
s^0 & K & 0 & 0
\end{array}$$

Where

$$c_1 = \frac{\epsilon - K}{\epsilon} \longrightarrow \frac{-K}{\epsilon}$$

- Therefore, for any value of *K* greater than zero, the system is unstable.
- Also, because the last term in the first column is equal to K, a negative value of K will result in an unstable system.
- Consequently, the system is unstable for all values of gain K.

Example-8: Determine the stability of the of the closed-loop transfer function;

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

Table-1: The complete Routh table is formed by using the denominator of the characteristic equation T(s).

s^5	1	3	5
s^4	2	6	3
s^3	$\cancel{0}\ \epsilon$	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

Table-2: shows the first column of Table-1 along with the resulting signs for choices of ε positive and ε negative.

Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	$\cancel{0}\ \epsilon$	+	−
s^2	$\frac{6\epsilon - 7}{\epsilon}$	−	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

- A zero appears only in the first column (the s3 row).
- Next replace the zero by a small number, ε, and complete the table.
- Assume a sign, positive or negative, for the quantity ε.
- When quantity ε is either positive or negative, in both cases the sign in the first column of Routh table is changes twice.
- Hence, **the system is unstable and has two poles in the right half-plane**.

Case-II: Stability via Reverse Coefficients (Phillips, 1991).

- A polynomial that has the reciprocal roots of the original polynomial has its roots distributed the same—right half-plane, left half plane, or imaginary axis—because taking the reciprocal of the root value does not move it to another region.
- If we can find the polynomial that has the reciprocal roots of the original, it is possible that the Routh table for the new polynomial will not have a zero in the first column.
- The polynomial with reciprocal roots is a polynomial with the coefficients written in reverse order.
- This method is usually computationally easier than the epsilon method.

Example-9: Repeated example-8: Determine the stability of the closed-loop transfer function;

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

- First write a polynomial that has the reciprocal roots of the denominator of *T(s).*
- This polynomial is formed by writing the denominator of *T(s)* in reverse order. Hence,

$$D(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1$$

- The Routh table is

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	−1.75		
s^0	1		

- **Since there are TWO sign changes, the system is unstable and has TWO right-half-plane poles.**
- This is the same as the result obtained in the previous Example.
- Notice that Table does not have a zero in the first column.

Case-III: Entire Row is Zero.

- Sometimes while making a Routh table, we find that an entire row consists of zeros.
- This happen because there is an even polynomial that is a factor of the original polynomial.
- This case must be handled differently from the case of a zero in only the first column of a row.

Example-10: Determine the stability of the system.

The characteristic equation q(s) of the system is $q(s) = s^3 + 2s^2 + 4s + K$

Where ***K*** is an adjustable loop gain. The Routh array is then;

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & K \\ s^1 & \dfrac{8-K}{2} & 0 \\ s^0 & K & 0 \end{array}$$

For a stable system, the value of ***K*** must be; $0 < K < 8$

When ***K*** = 8, the two roots exist on the *j*ω axis and the system will be marginally stable.

- Also, **when *K* = 8, we obtain a row of zeros (case-III).**
- The **auxiliary polynomial, *U(s)*,** is the equation of the row preceding the row of Zeros.
- The ***U(s)*** in this case, obtained from the $\mathbf{s^2}$ row.
- The order of the auxiliary polynomial is **always even** and indicates the number of **symmetrical root pairs**.

Example-10: continue.

$$q(s) = s^3 + 2s^2 + 4s + K$$

- The auxiliary polynomial, ***U(s)***, can be obtain as;

$$\begin{aligned} U(s) &= 2s^2 + Ks^0 \\ &= 2s^2 + 8 \\ &= 2(s^2 + 4) \\ &= 2(s + j2)(s - j2) \end{aligned}$$

- To show that the auxiliary polynomial, ***U(s)***, is indeed a factor of the characteristic polynomial, q(s), we divide q(s) by U(s) to obtain

$$\begin{array}{r l} & \frac{1}{2}s + 1 \\ 2s^2 + 8 \big) & \overline{s^3 + 2s^2 + 4s + 8} \\ & \underline{s^3 \qquad\quad + 4s} \\ & \qquad 2s^2 \qquad\quad + 8 \\ & \qquad 2s^2 \qquad\quad + 8 \end{array}$$

- When ***K* = 8**, the factors of the characteristic polynomial, q(s), are

$$q(s) = (s + 2)(s + j2)(s - j2)$$

Case-IV: Repeated roots of the characteristic equation on the *jω*-axis.

- If the ***jω*-axis roots are repeated**, the system response will be **unstable** with a form $t\sin(\omega t + \Phi)$. The Routh-Hurwitz criteria will not reveal this form of instability.

Example-11: Determine the stability of the system with the characteristic equation of

$$q(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63$$

The Routh array is;

$$\begin{array}{c|rrr}
s^5 & 1 & 4 & 3 \\
s^4 & 1 & 24 & 63 \\
s^3 & -20 & -60 & 0 \\
s^2 & 21 & 63 & 0 \\
s^1 & 0 & 0 & 0
\end{array}$$

Example-11:Continue.

Therefore, the Auxiliary polynomial, ***U(s)***, is;

$$U(s) = 21s^2 + 63 = 21(s^2 + 3) = 21\left(s + j\sqrt{3}\right)\left(s - j\sqrt{3}\right)$$

Which indicates that **TWO roots are on the imaginary *jω*-axis**.

To examine the remaining roots, we divide the characteristic equation, ***q(s)***, by the auxiliary polynomial, ***U(s),*** to obtain;

$$\frac{q(s)}{s^2 + 3} = s^3 + s^2 + s + 21 \longrightarrow \text{(a)}$$

Establishing a Routh table for this equation, we have;

s^3	1	1
s^2	1	21
s^1	−20	0
s^0	21	0

- The **TWO** changes in sign in the first column indicate the presence of **TWO** roots in the right-hand plane, and the **system is unstable**.
- There are **THREE** roots of eq. (a). The **ONE** root in left-hand side is s = - 3.
- The **TWO** roots in the right-hand plane are $s = +1 \pm j\sqrt{6}$.

Skill Assessment # 3:

Problem: Make a Routh table and tell how many roots of the following polynomial are in the right half-plane and in the left half-plane.

$$P(s) = 3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6$$

Answer of the Skill Assessment # 3:

- **Four roots** of the characteristic equation lie in the right half-plane (rhp), and **three roots** lie of the characteristic equation in the left half-plane (lhp). Therefore the **system is unstable**.