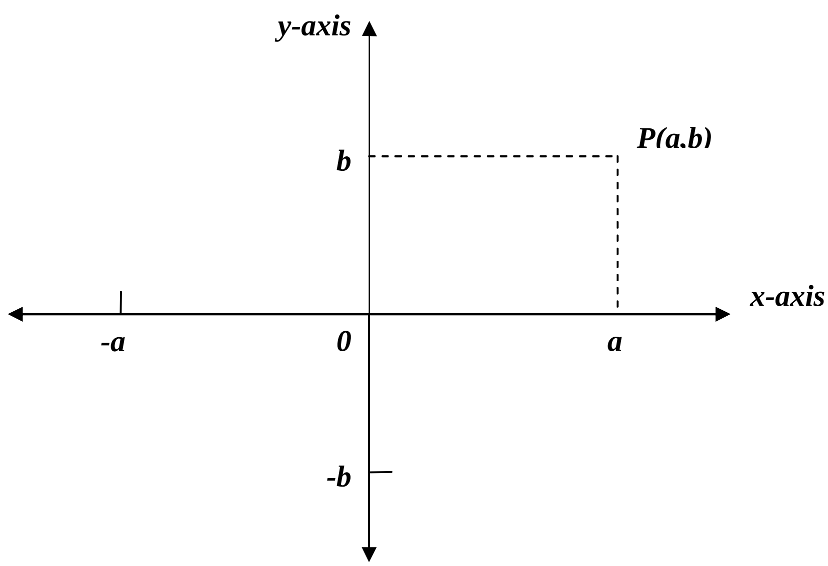
Chapter – one
The Rate of Change of a Function

1-1- Coordinates for the plane :

***Cartesian Coordinate*- Two number lines , one of them horizontal (called *x-axis*) and the other vertical (called *y-axis*). The point where the lines cross is the *origin* . Each line is assumed to represent the real number .**

On the *x-axis* , the positive number *a* lies *a* units to the right of the *origin* , and the negative number *–a* lies *a* units to the left of the *origin* . On the *y-axis* , the positive number *b* lies *b* units above the *origin* while the negative where *–b* lies *b* units below the *origin* .

With the axes in place , we assign a pair *(a,b)* of real number to each point *P* in the plane . The number *a* is the number at the foot of the perpendicular from *P* to the *x-axis* (called *x-coordinate of P*). The number *b* is the number at the foot of the perpendicular from *P* to the *y-axis* (called *y-coordinate of P*).



1-2- The Slope of a line :

***Increments* – When a particle moves from one position in the plane to another , the net changes in the particle's coordinates are calculated by subtracting the coordinates of the starting point (x_1 , y_1) from the coordinates of the stopping point (x_2 , y_2) ,**

i.e. $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$.

***Slopes of nonvertical lines* :**

Let L be a nonvertical line in the plane ,

Let $P_1(x_1 , y_1)$ and $P_2 (x_2 , y_2)$ be two points on L.

Then the slope m is :

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \qquad where \quad \Delta x \neq 0$$

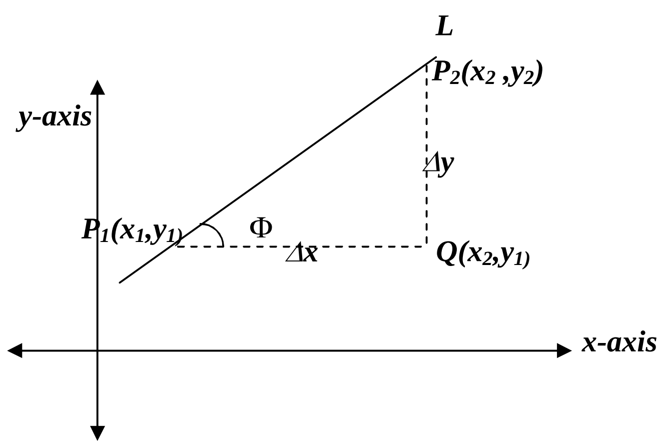
- A line that goes uphill as x increases has a positive slope . A line that goes downhill as x increases has a negative slope .
- A horizontal line has slope zero because $\Delta y = 0$.
- The slope of a vertical line is undefined because $\Delta x = 0$.
- Parallel lines have same slope .
- If neither of two perpendicular lines L_1 and L_2 is vertical , their slopes m_1 and m_2 are related by the equation : $m_1 . m_2 = -1$.

Angles of Inclination: The angle of inclination of a line that crosses the *x-axis* is the smallest angle we get when we measure counter clock from the *x-axis* around the point of intersection .

The slope of a line is the tangent of the line angle of inclination .

$m = \tan \Phi$ where Φ is the angle of inclination .

- The angle of inclination of a horizontal line is taken to be 0^{o}.
- Parallel lines have equal angle of inclination .



EX-1- Find the slope of the line determined by two points $A(2,1)$ and $B(-1,3)$ and find the common slope of the line perpendicular to AB.

Sol.- Slope of AB is: $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{-1-2} = -\frac{2}{3}$

Slope of line perpendicular to AB is : $-\frac{1}{m_{AB}} = \frac{3}{2}$

EX-2- Use slopes to determine in each case whether the points are collinear (lie on a common straight line) :

a) $A(1,0)$, $B(0,1)$, $C(2,1)$.

b) $A(-3,-2)$, $B(-2,0)$, $C(-1,2)$, $D(1,6)$.

Sol. –

a) $m_{AB} = \frac{1-0}{0-1} = -1 \quad and \quad m_{BC} = \frac{1-1}{2-0} = 0 \neq m_{AB}$

The points A , B and C are not lie on a common straight line .

b) $m_{AB} = \frac{0-(-2)}{-2-(-3)} = 2 \quad , \quad m_{BC} = \frac{2-0}{-1-(-2)} = 2 \quad , \quad m_{CD} = \frac{6-2}{1-(-1)} = 2$

Since $m_{AB} = m_{BC} = m_{CD}$

Hence the points A , B , C , and D are collinear .

1-3- Equations for lines : An equation for a line is an equation that is satisfied by the coordinates of the points that lies on the line and is not satisfied by the coordinates of the points that lie elsewhere .

Vertical lines : Every vertical line L has to cross the x-axis at some point $(a,0)$. The other points on L lie directly above or below $(a,0)$. This mean that : $x = a \quad \forall (x, y)$

Nonvertical lines : That point – slope equation of the line through the point (x_1 , y_1) with slope m is :

$$y - y_1 = m\,(x - x_1)$$

Horizontal lines : The standard equation for the horizontal line through the point (a , b) is : $y = b$.

The distance from a point to a line : To calculate the distance d between the point $P(x_1 , y_1)$ and $Q(x_2 , y_2)$ is :

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We use this formula when the coordinate axes are scaled in a common unit .

To find the distance from the point $P(x_1 , y_1)$ to the line L , we follow :

1. Find an equation for the line L' through P perpendicular to L :

   $y - y_1 = m'\,(x - x_1)$ where $m' = -1/m$

2. Find the point $Q(x_2 , y_2)$ by solving the equation for L and L' simultaneously .
3. Calculate the distance between P and Q .

The general linear equation :

$$Ax + By = C$$ where A and B not both zero.

EX-3 – Write an equation for the line that passes through point :

a) $P(-1 , 3)$ with slope $m = -2$.

b) $P_1(-2 , 0)$ and $P_2(2 , -2)$.

Sol. - a) $y - y_1 = m\,(x - x_1) \rightarrow y - 3 = -2\,(x - (-1)) \rightarrow y + 2x = 1$

b)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-0}{2-(-2)} = -\frac{1}{2}$$

$$y - y_1 = m(x - x_1) \Rightarrow \quad y - 0 = -\frac{1}{2}(x - (-2)) \Rightarrow \quad 2y + x + 2 = 0$$

EX-4 - **Find the slope of the line :** $3x + 4y = 12$ **.**

Sol. - $y = -\frac{3}{4}x + 3 \Rightarrow \quad the \quad slope \quad is \quad m = -\frac{3}{4}$

EX-5- **Find :**

a) **an equation for the line through** $P(2,1)$ **parallel to** $L: \ y = x + 2$ **.**

b) **an equation for the line through** P **perpendicular to** L **.**

c) **the distance from** P **to** L **.**

Sol.-

a)

$$since \quad L_2 // L_1 \Rightarrow \quad m_{L2} = m_{L1} = 1 \Rightarrow \quad y - 1 = 1(x - 2) \quad \Rightarrow \quad y = x - 1$$

b) Since L_1 **and** L_3 **are perpendicular lines then :**

$$m_{L3} = -1 \Rightarrow \quad y - 1 = -(x - 2) \Rightarrow \quad y + x = 3$$

c)

$$\begin{matrix} y = x + 2 \\ y + x = 3 \end{matrix} \quad \Rightarrow \quad x = \frac{1}{2} \quad and \quad y = \frac{5}{2} \quad \Rightarrow \quad P(2,1) \quad and \quad Q\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$\Rightarrow d = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} = \sqrt{4.5}$$

EX-6 **– Find the angle of inclination of the line :** $\sqrt{3}x + y = -3$

Sol.-

$$y = -\sqrt{3}x - 3 \quad \Rightarrow \quad m = -\sqrt{3}$$

$$m = tan\Phi = -\sqrt{3} \quad \Rightarrow \quad \Phi = 120^o$$

EX-7- **Find the line through the point** $P(1, 4)$ **with the angle of inclination** $\Phi = 60^o$ **.**

Sol.-

$$m = tan\Phi = tan 60 = \sqrt{3}$$

$$y - 4 = \sqrt{3}(x - 1) \Rightarrow \quad y = \sqrt{3}x + 4 - \sqrt{3}$$

EX-8- **The pressure** P **experienced by a diver under water is related to the diver's depth** d **by an equation of the form** $P = k\,d + 1$ **where** k **a constant . When** $d = 0$ **meters , the pressure is** 1 **atmosphere . The pressure at** 100 **meters is about** 10.94 **atmosphere . Find the pressure at** 50 **meters.**

Sol.- **At** $P = 10.94$ **and** $d = 100 \ \rightarrow \ 10.94 = k(100)+1 \rightarrow \ k = 0.0994$

$P = 0.0994\,d + 1$ **, at** $d = 50 \ \rightarrow \ P = 0.0994 * 50 + 1 = 5.97$ **atmo.**

1-4- <u>Functions</u> : ***Function*** **is any rule that assigns to each element in one set some element from another set :**

$$y = f(x)$$

The inputs make up the *domain of the function* , and the outputs make up *the function's range.*

The variable x is called *independent variable of the function* , and the variable y whose value depends on x is called *the dependent variable of the function* .

We must keep two restrictions in mind when we define functions :

1. **We never divide by zero .**
2. **We will deal with real – valued functions only.**

<u>Intervals</u> :

- **The *open interval* is the set of all real numbers that be strictly between two fixed numbers a and b :**

$$(a,b) \equiv a < x < b$$

- **The *closed interval* is the set of all real numbers that contain both endpoints :**

$$[a,b] \equiv a \le x \le b$$

- ***Half open interval* is the set of all real numbers that contain one endpoint but not both :**

$$[a,b) \equiv a \le x < b$$

$$(a,b] \equiv a < x \le b$$

<u>Composition of functions</u> : **suppose that the outputs of a function f can be used as inputs of a function g . We can then hook f and g together to form a new function whose inputs are the inputs of f and whose outputs are the numbers :**

$$(g_o f)(x) = g(f(x))$$

***<u>EX-9</u>*- Find the domain and range of each function :**

$$a)\quad y = \sqrt{x+4} \quad , \quad b)\quad y = \frac{1}{x-2}$$

$$c)\quad y = \sqrt{9-x^2} \quad , \quad d)\quad y = \sqrt{2-\sqrt{x}}$$

<u>Sol.</u> - $a)\quad x+4 \ge 0 \Rightarrow \quad x \ge -4 \Rightarrow \quad D_x : \forall x \ge -4 \quad , \quad R_y : \forall y \ge 0$

$$b)\quad x-2 \ne 0 \Rightarrow \quad x \ne 2 \Rightarrow \quad D_x : \forall x \ne 2$$

$$y = \frac{1}{x-2} \Rightarrow \quad x = \frac{1}{y} + 2 \Rightarrow \quad R_y : \forall y \ne 0$$

$$c)\quad 9 - x^2 \ge 0 \Rightarrow \quad -3 \le x \le 3 \Rightarrow \quad D_x : -3 \le x \le 3$$

$$y = \sqrt{9-x^2} \Rightarrow \quad x = \mp\sqrt{9-y^2}$$

$$since \quad 9 - y^2 \ge 0 \Rightarrow \quad -3 \le y \le 3$$

$$since \quad y \ge 0 \quad \Rightarrow \quad R_y : 0 \le y \le 3$$

$$d)\quad 2-\sqrt{x}\ge 0 \Rightarrow \quad 0\le x\le 4 \Rightarrow \quad D_x : 0\le x\le 4$$

$$\begin{array}{lll} if & x=0\Rightarrow & y=\sqrt{2} \\ if & x=4\Rightarrow & y=0 \end{array} \quad \Rightarrow \quad R_y : 0\le y\le \sqrt{2}$$

EX-10- Let $f(x)=\dfrac{x}{x-1}$ and $g(x)=1+\dfrac{1}{x}$.

Find $(g_o f)(x)$ and $(f_o g)(x)$.

Sol.-

$$(g_o f)(x)=g(f(x))=g\left(\frac{x}{x-1}\right)=1+\frac{1}{\frac{x}{x-1}}=\frac{2x-1}{x}$$

$$(f_o g)(x)=f(g(x))=f\left(1+\frac{1}{x}\right)=\frac{1+\frac{1}{x}}{1+\frac{1}{x}-1}=x+1$$

EX-11- Let $(g_o f)(x)=x$ and $f(x)=\dfrac{1}{x}$. Find $g(x)$.

Sol.- $(g_o f)(x)=g\left(\dfrac{1}{x}\right)=x\Rightarrow \quad g(x)=\dfrac{1}{x}$

1-5- Limits and continuity :

Limits : The limit of $F(t)$ as t approaches C is the number L if : Given any radius $\varepsilon>0$ about L there exists a radius $\delta>0$ about C such that for all t, $0<|t-C|<\delta$ implies $|F(t)-L|<\varepsilon$ and we can write it as :

$$\lim_{t\to C} F(t)=L$$

The limit of a function $F(t)$ as $t\to C$ never depend on what happens when $t=C$.

Right hand limit : $\lim\limits_{t\to C^+} F(t)=L$

The limit of the function $F(t)$ as $t\to C$ from the right equals L if : Given any $\varepsilon>0$ (radius about L) there exists a $\delta>0$ (radius to the right of C) such that for all t :

$$C<t<C+\delta \Rightarrow \quad |F(t)-L|<\varepsilon$$

Left hand limit : $\lim\limits_{t\to C^-} F(t)=L$

The limit of the function $F(t)$ as $t\to C$ from the left equal L if : Given any $\varepsilon>0$ there exists a $\delta>0$ such that for all t :

$$C-\delta<t<C\Rightarrow \quad |F(t)-L|<\varepsilon$$