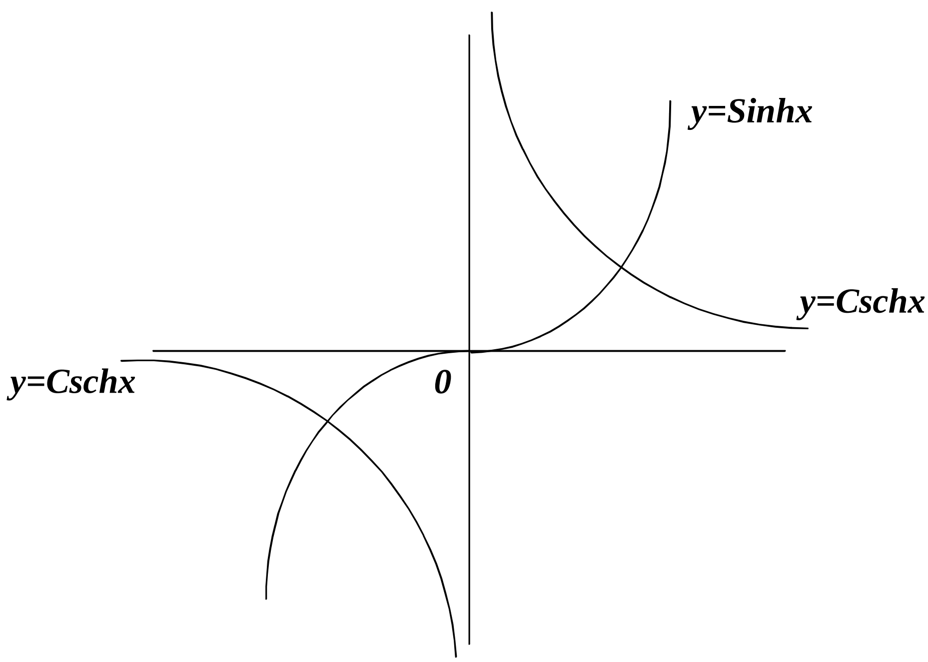
2-4- <u>Hyperbolic functions</u> : Hyperbolic functions are used to describe the motions of waves in elastic solids ; the shapes of electric power lines ; temperature distributions in metal fins that cool pipes …etc.

The hyperbolic sine (Sinh) and hyperbolic cosine (Cosh) are defined by the following equations :

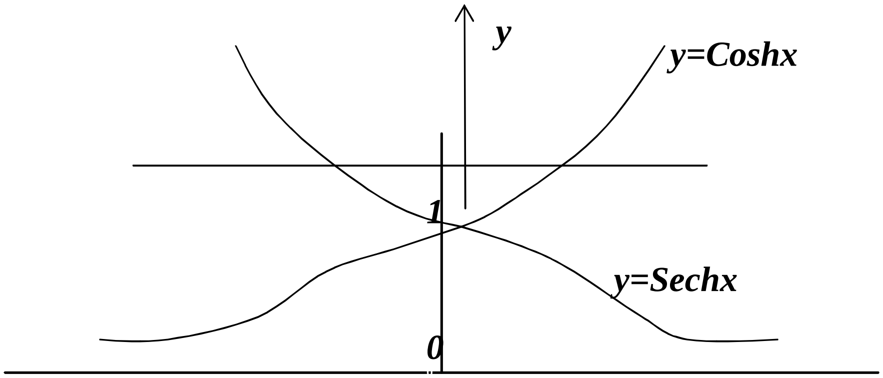
1. $Sinhu = \frac{1}{2}(e^u - e^{-u}) \quad and \quad Coshu = \frac{1}{2}(e^u + e^{-u})$
2. $\tanh u = \frac{Sinhu}{Coshu} = \frac{e^u - e^{-u}}{e^u + e^{-u}} \quad and \quad Cothu = \frac{Coshu}{Sinhu} = \frac{e^u + e^{-u}}{e^u - e^{-u}}$
3. $Sechu = \frac{1}{Coshu} = \frac{2}{e^u + e^{-u}} \quad and \quad Cschu = \frac{1}{Sinhu} = \frac{2}{e^u - e^{-u}}$

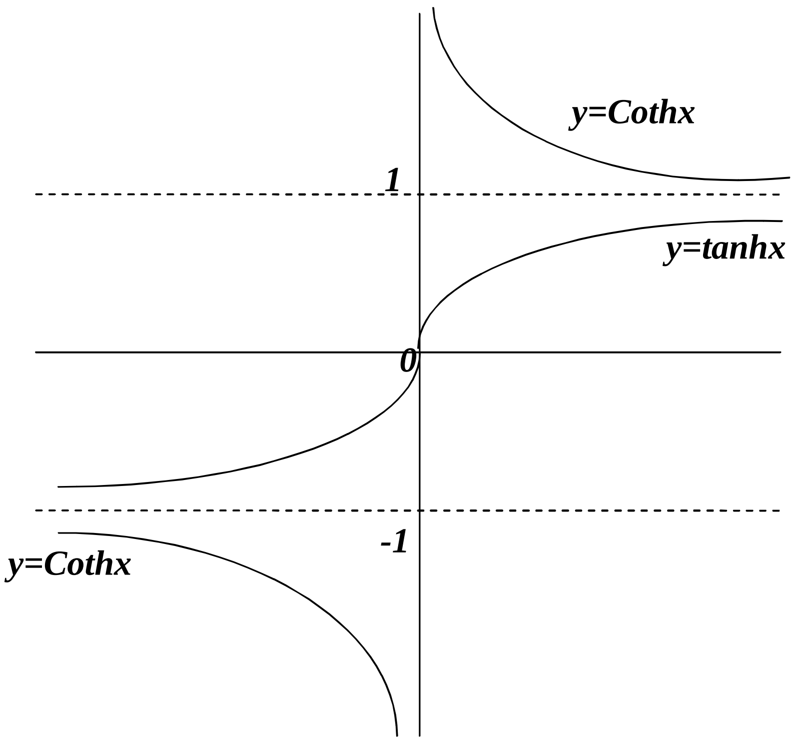
Asso. Prof.Dr.Mahdi Hatf Kadhum

4. $Cosh^2u - Sinh^2u = 1$
5. $\tanh^2 u + Sech^2u = 1 \quad and \quad Coth^2u - Csch^2u = 1$
6. $Coshu + Sinhu = e^u \quad and \quad Coshu - Sinhu = e^{-u}$
7. $Cosh(-u) = Coshu \quad and \quad Sinh(-u) = -Sinhu$
8. $Cosh0 = 1 \quad and \quad Sinh0 = 0$
9. $Sinh(x + y) = Sinhx.Coshy + Coshx.Sinhy$
10. $Cosh(x + y) = Coshx.Coshy + Sinhx.Sinhy$
11. $Sinh2x = 2.Sinhx.Coshx$
12. $Cosh2x = Cosh^2 x + Sinh^2 x$
13. $Cosh^2 x = \frac{Cosh2x + 1}{2} \quad and \quad Sinh^2 x = \frac{Cosh2x - 1}{2}$



Sol.-





$$y = Sinhx \quad D_x : \forall x \quad and \quad R_y : \forall y$$
$$y = Coshx \quad D_x : \forall x \quad and \quad R_y : y \geq 1$$
$$y = \tanh x \quad D_x : \forall x \quad and \quad R_y : -1 \leq y \leq 1$$
$$y = Cothx \quad D_x : \forall x \neq 0 \quad and \quad R_y : y < -1 \;\; or \;\; y > 1$$
$$y = Sechx \quad D_x : \forall x \quad and \quad R_y : 0 < y \leq 1$$
$$y = Cschx \quad D_x : \forall x \neq 0 \quad and \quad R_y : \forall y \neq 0$$

EX-16- **Let *tanh u = - 7 / 25* , determine the values of the remaining five hyperbolic functions .**

Sol.-

$$Cothu=\frac{1}{\tanh u}=-\frac{25}{7}$$

$$\tanh^2 u+Sech^2 u=1 \quad \Rightarrow \frac{49}{625}+Sech^2 u=1\Rightarrow Sechu=\frac{24}{25}$$

$$Coshu=\frac{1}{Sechu}=\frac{25}{24}$$

$$\tanh u=\frac{Sinhu}{Coshu} \qquad \Rightarrow -\frac{7}{25}=\frac{Sinhu}{\frac{25}{24}}\Rightarrow Sinhu=-\frac{7}{24}$$

$$Cschu=\frac{1}{Sinhu}=-\frac{24}{7}$$

***EX-17*-** **Rewrite the following expressions in terms of exponentials . Write the final result as simply as you can :**

$a)\ 2Cosh(\ln x) \qquad b)\ \tanh(\ln x)$

$c)\ Cosh5x+Sinh5x \quad d)\ (Sinhx+Coshx)^4$

***Sol.*-**

$$a)\quad 2Cosh(\ln x)=2.\frac{e^{\ln x}+e^{-\ln x}}{2}=x+\frac{1}{x}$$

$$b)\quad \tanh(\ln x)=\frac{e^{\ln x}-e^{-\ln x}}{e^{\ln x}+e^{-\ln x}}=\frac{x-\frac{1}{x}}{x+\frac{1}{x}}=\frac{x^2-1}{x^2+1}$$

$$c)\quad Cosh5x+Sinh5x=\frac{e^{5x}+e^{-5x}}{2}+\frac{e^{5x}-e^{-5x}}{2}=e^{5x}$$

$$d)\quad (Sinhx+Coshx)^4=\left(\frac{e^x-e^{-x}}{2}+\frac{e^x+e^{-x}}{2}\right)^4=e^{4x}$$

EX-18*-** **Solve the equation for *x* : *Cosh x = Sinh x + 1 / 2 .

Sol. - $Coshx-Sinhx=\frac{1}{2}\Rightarrow e^{-x}=\frac{1}{2}\Rightarrow -x=\ln 1-\ln 2\Rightarrow x=\ln 2$

EX-19 **– Verify the following identity :**

a) ***Sinh(u+v)=Sinh u. Cosh v + Cosh u.Sinh v***

b) then verify ***Sinh(u-v)=Sinh u. Cosh v - Cosh u.Sinh v***

***Sol.*-**

$$a)\ R.H.S. = Sinhu.Coshv + Coshu.Sinhv$$
$$= \frac{e^u - e^{-u}}{2}.\frac{e^v + e^{-v}}{2} + \frac{e^u + e^{-u}}{2}\frac{e^v - e^{-v}}{2}$$
$$= \frac{e^{u+v} - e^{-(u+v)}}{2} = Sinh(u+v) = L.H.S.$$
$$b)\ L.H.S. = Sinh(u+(-v)) = Sinhu.Cosh(-v) + Coshu.Sinh(-v)$$
$$= Sinhu.Coshv - Coshu.Sinhv = R.H.S.$$

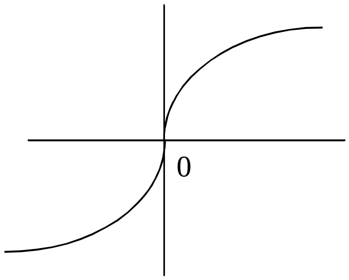
<u>*EX-20*</u> – Verify the following identities :

$$a)\quad Sinhu.Coshv = \frac{1}{2}\left[Sinh(u+v) + Sinh(u-v)\right]$$
$$b)\quad Coshu.Coshv = \frac{1}{2}\left[Cosh(u+v) + Cosh(u-v)\right]$$
$$c)\quad Sinh3u = Sinh^3u + 3Cosh^2u.Sinhu = 3Sinhu + 4Sinh^3u$$
$$d)\quad Sinh^2u - Sinh^2v = Cosh^2u - Cosh^2v$$

<u>Sol.</u> –

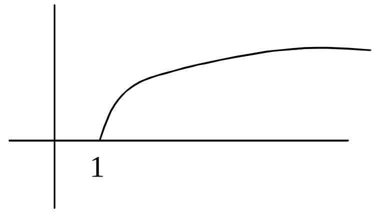
$$a)\ R.H.S. = \frac{1}{2}\left[Sinh(u+v) + Sinh(u-v)\right]$$
$$= \frac{1}{2}\left[Sinhu.Coshv + Coshu.Sinhv + Sinhu.Coshv - Coshu.Sinhv\right]$$
$$= Sinhu.Coshv = L.H.S.$$
$$b)\ R.H.S. = \frac{1}{2}\left[Cosh(u+v) + Cosh(u-v)\right]$$
$$= \frac{1}{2}\left[Coshu.Coshv + Sinhu.Sinhv + Coshu.Coshv - Sinhu.Sinhv\right]$$
$$= Coshu.Coshv = L.H.S.$$
$$c)\ L.H.S. = Sinh(2u+u) = Sinh2u.Coshu + Cosh2u.Sinhu$$
$$= 2Sinhu.Coshu.Coshu + (Cosh^2u + Sinh^2u).Sinhu$$
$$= 3Sinhu.Cosh^2u + Sinh^3u = R.H.S.(I)$$
$$= 3Sinhu.(1 + Sinh^2u) + Sinh^3u = 3Sinhu + 4Sinh^3u = R.H.S.(II)$$
$$d)\ L.H.S. = Sinh^2u - Sinh^2v = Cosh^2u - 1 - (Cosh^2v - 1)$$
$$= Cosh^2u - Cosh^2v = R.H.S.$$

2-5- <u>Inverse hyperbolic functions</u> : All hyperbolic functions have inverses that are useful in integration and interesting as differentiable functions in their own right .



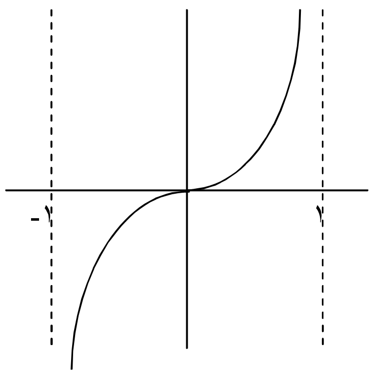
$y = Sinh^{-1}x \quad D_x : \forall x$

$R_y : \forall y$



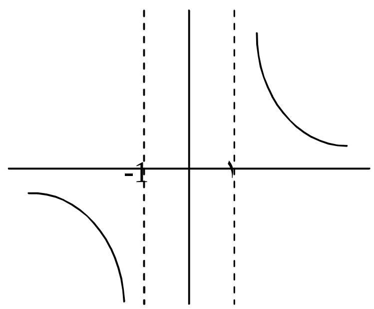
$y = Cosh^{-1}x \quad D_x : \forall x \geq 1$

$R_y : \forall y \geq 0$



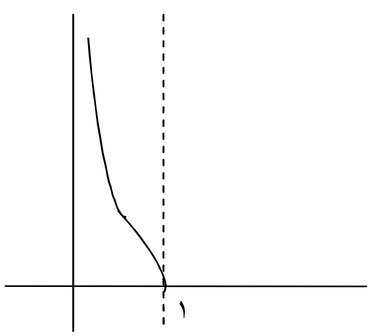
$y = tanh^{-1}x \quad D_x : -1 < x < 1$

$R_y : \forall y$



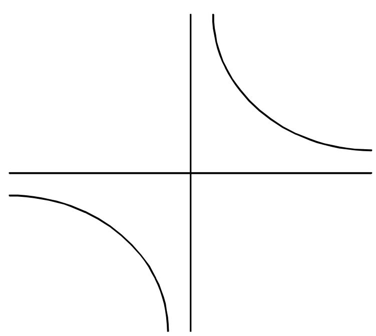
$y = Coth^{-1}x \quad D_x : \forall x < -1 \text{ or } x > 1$

$R_y : \forall y \neq 0$



$y = Sech^{-1}x \quad D_x : 0 < x \leq 1$

$R_y : \forall y \geq 0$



$y = Csch^{-1}x \quad D_x : \forall x \neq 0$

$R_y : \forall y \neq 0$

Some useful identities :

1. $Sinh^{-1}x = ln(\,x+\sqrt{x^2+1}\,)$
2. $Cosh^{-1}x = ln(\,x+\sqrt{x^2-1}\,)$
3. $tanh^{-1}\,x = \dfrac{1}{2}.ln\left(\dfrac{1+x}{1-x}\right)$
4. $Coth^{-1}x = \dfrac{1}{2}.ln\left(\dfrac{x+1}{x-1}\right) = tanh^{-1}\,\dfrac{1}{x}$
5. $Sech^{-1}x = ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right) = Cosh^{-1}\,\dfrac{1}{x}$
6. $Csch^{-1}x = ln\left(\dfrac{1}{x}+\dfrac{\sqrt{x^2+1}}{|x|}\right) = Sinh^{-1}\,\dfrac{1}{x}$

EX-21 **- Derive the formula :**

$$Sinh^{-1}x = ln(\,x+\sqrt{x^2+1}\,)$$

Sol.-

$$Let\;\; y = Sinh^{-1}x \Rightarrow x = Sinhy = \frac{e^y - e^{-y}}{2} \Rightarrow x = \frac{e^{2y}-1}{2e^y}$$

$$\Rightarrow e^{2y} - 2x.e^y - 1 = 0$$

$$e^y = \frac{2x \mp \sqrt{4x^2+4}}{2} \Rightarrow e^y = x \mp \sqrt{x^2+1}$$

$$either\;\; y = ln(\,x-\sqrt{x^2+1}\,) \qquad neglected\;\; since\;\; x-\sqrt{x^2+1} < 0$$

$$or\qquad y = ln(\,x+\sqrt{x^2+1}\,)$$

Problems – 2

1. A body of unknown temperature was placed in a room that was held at 30^o *F* . After *10* minutes , the body's temperature was 0^oF , and *20* minutes after the body was placed in the room the body's temperature 15^oF . Use Newton's law of cooling to estimate the body's initial temperature . *(ans.:-30^oF)*

2. A pan of warm water 46^oC was put in a refrigerator . Ten minutes later , the water's temperature was 39^oC , *10* minutes after that , it was 33^oC . Use Newton's law of cooling to estimate how cold the refrigerator was ? *(ans.:-3^oC)*

3. Solve the following equations for values of θ from -180^o to 180^o inclusive:
   i) $\tan^2\theta + \tan\theta = 0$ ii) $\cot\theta = 5\cos\theta$
   iii) $3\cos\theta + 2\sec\theta + 7 = 0$ iv) $\cos^2\theta + \sin\theta + 1 = 0$
   (ans.:i)-180,-45,0,135,180; ii)-90,11.5,90,168.5; iii)-109.5,109.5; iv)-90)

4. Solve the following equations for values of θ from 0^o to 360^o inclusive:
   i) $3\cos 2\theta - \sin\theta + 2 = 0$ ii) $3\tan\theta = \tan 2\theta$
   iii) $\sin 2\theta . \cos\theta + \sin^2\theta = 1$ iv) $3\cot 2\theta + \cot\theta = 1$
   (ans.:i)56.4,123.6,270; ii)0,30,150,180,210,330,360; iii)30,90,150,270; iv)45,121,225,301)

5. If $\sin\theta = 3/5$, find without using tables the values of :
   i) $\cos\theta$ ii) $\tan\theta$ *(ans.: i) 4/5 ; ii) 3/4)*

6. Find, without using tables, the values of $\cos x$ and $\sin x$, when $\cos 2x$ is :
   a) 1/8 , b) 7/25 , c) -119/169

$$(ans.: a)\mp\frac{3}{4},\mp\frac{\sqrt{7}}{4};b)\mp\frac{4}{5},\mp\frac{3}{5};c)\mp\frac{5}{13},\mp\frac{12}{13})$$

7. If $\sin A = 3/5$ and $\sin B = 5/13$, where *A* and *B* are acute angles , find without using tables , the values of :
   a) $\sin(A+B)$, b) $\cos(A+B)$, c) $\cot(A+B)$ *(ans.: 56/65; 33/65; 33/56)*

8. If $\tan A = -1/7$ and $\tan B = 3/4$, where *A* is obtuse and *B* is acute , find without using tables the value of $A - B$. *(ans.: 135)*

9. Prove the following identities :

$i)\quad Sec^2\theta + Csc^2\theta = Sec^2\theta . Csc^2\theta$

$ii)\quad Sin^2\theta(1+Sec^2\theta) = Sec^2\theta - Cos^2\theta$

$iii)\quad \dfrac{1+Sin\theta}{1-Sin\theta} = (Sec\theta + tan\theta)^2$

$iv)\quad Sec\theta - Sin\theta = \dfrac{tan^2\theta + Cos^2\theta}{Sec\theta + Sin\theta}$

$v)\quad \dfrac{Cos(A-B) - Cos(A+B)}{Sin(A+B) + Sin(A-B)} = tan B$

$vi)\quad CosB - CosA.Cos(A-B) = SinA.Sin(A-B)$

$vii)\quad tan(A+B+C) = \dfrac{tan A + tan B + tan C - tan A.tan B.tan C}{1 - tan B.tan C - tan C.tan A - tan A.tan B}$

If A,B,C are angles of a triangle, show that :

$tan A + tan B + tan C = tan A.tan B.tan C$

$viii)\quad \dfrac{1}{2}\left[tan(x+h) + tan(x-h)\right] - tan x = \dfrac{tan x.Sin^2 h}{Cos^2 x - Sin^2 h}$

$ix)\quad tan x = \sqrt{\dfrac{1-Cos2x}{1+Cos2x}}$

$x)\quad \dfrac{Sin4A + Sin2A}{Cos4A + Cos2A + 1} = tan 2A$

$xi)\quad Sin^4\theta + Cos^4\theta = \dfrac{1}{4}(Cos4\theta + 3)$

$xii)\quad 4Sin^3A.Cos3A + 4Cos^3A.Sin3A = 3Sin4A$

$xiii)\quad tan 3A = \dfrac{3tan A - tan^3 A}{1 - 3tan^2 A}$

$xiv)\quad Cos^{-1}(-x) = \pi - Cos^{-1}x$

$xv)\quad Cot^{-1}x = \dfrac{\pi}{2} - tan^{-1}x$

$xvi)\quad Cosh(u+v) = Coshu.Coshv + Sinhu.Sinhv$

and then verify

$Cosh(u-v) = Coshu.Coshv - Sinhu.Sinhv$

$xvii)\quad Coshu.Sinhv = \dfrac{1}{2}\left[Sinh(u+v) - Sinh(u-v)\right]$

$xviii)\quad Sinhu.Sinhv = \dfrac{1}{2}\left[Cosh(u+v) - Cosh(u-v)\right]$

$xix)\quad Cosh3u = Coshu + 4Sinh^2u.Coshu = 4Cosh^3u - 3Coshu$

$xx)\quad (Coshx + Sinhx)^n = Coshnx + Sinhnx$

10. If $u = \dfrac{1+Sin\theta}{Cos\theta}$, prove that $\dfrac{1}{u} = \dfrac{1-Sin\theta}{Cos\theta}$ and deduce formula for $Sin\theta$, $Cos\theta$, $tan\theta$ in terms of u. *(ans.:$(u^2-1)/(u^2+1)$; $2u/(u^2+1)$;$(u^2-1)/(u^2+1)$)*

11. If $Sin(x+\alpha)=2Cos(x-\alpha)$; prove that : $tan\,x=\dfrac{2-tan\alpha}{1-2tan\alpha}$.

12. If $Sin(x-\alpha)=Cos(x+\alpha)$; prove that : $tan\,x=1$.

13. If $x=Cos\theta+Cos2\theta$ and $y=Sin\theta+Sin2\theta$. Show that :

i) $x^2-y^2=Cos2\theta+2Cos3\theta+Cos4\theta$

ii) $2xy=Sin2\theta+2Sin3\theta+Sin4\theta$

14. If $Cos2A.Cos2B=Cos2\theta$, prove that :

$Sin^2A.Cos^2B+Cos^2A.Sin^2B=Sin^2\theta$

15. If $S = Sin\,\theta$ and $C = Cos\,\theta$, simplify :

i) $\dfrac{S.C}{\sqrt{1-S^2}}$, ii) $\dfrac{S.\sqrt{1-S^2}}{C.\sqrt{1-C^2}}$, iii) $\dfrac{C}{S}+\dfrac{S}{C}$

(ans.:i) $Sin\theta$; ii)1; iii) $Sec\theta.Csc\theta$)

16. Eliminate θ from the following equations :

i) $x=a.Csc\theta$ and $y=b.Sec\theta$

ii) $x=Sin\theta+Cos\theta$ and $y=Sin\theta-Cos\theta$

iii) $x=Sin\theta+tan\theta$ and $y=Sin\theta-tan\theta$

iv) $x=tan\theta$ and $y=tan2\theta$

(ans: i) $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=1$; ii) $x^2+y^2=2$; iii) $\dfrac{4}{(x+y)^2}-\dfrac{4}{(x-y)^2}=1$; iv) $y=\dfrac{2x}{1-x^2}$)

17. In the acute – angled triangle *OPQ* , the altitude *OR* makes angles *A* and *B* with *OP* and *OQ* . Show by means of areas that if *OP=q* , *OQ=p* , *OR=r* : *p.q.Sin(A+B) = q.r.SinA + p.r. SinB.*

18. Given that $\alpha=Sin^{-1}\dfrac{1}{2}$, find $Cos\alpha$, $tan\alpha$, $Sec\alpha$, and $Csc\alpha$.

(ans.: $\dfrac{\sqrt{3}}{2};\dfrac{1}{\sqrt{3}};\dfrac{2}{\sqrt{3}};2$)

19. Evaluate the following expressions :

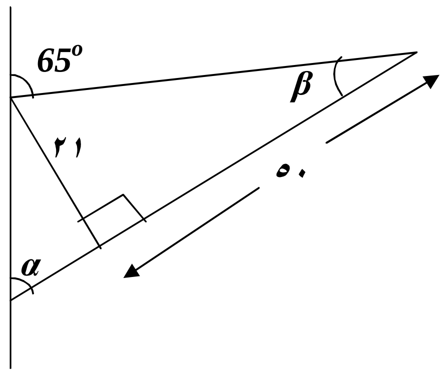
$a)\ Sin(Cos^{-1}\frac{1}{\sqrt{2}})$ $b)\ Csc(Sec^{-1}2)$

$c)\ Cot(Cos^{-1}0)$ $d)\ Sin^{-1}1 - Sin^{-1}(-1)$

$e)\ Cos(Sin^{-1}0.8)$ $f)\ Cos^{-1}(-Sin\frac{\pi}{6})$

$(ans.: 1/\sqrt{2}; 2/\sqrt{3}; 0; \pi; 0.6; 2\pi/3)$

20. Find the angle α in the below graph (Hint : $\alpha+\beta = 65^o$) :



(ans.: 42.2)

21. Let $Sech\ u = 3/5$, determine the values of the remaining five hyperbolic functions .

$(ans.: Coshu = 5/3; Sinhu = \mp 4/3; tanh\,u = \mp 4/5; Cothu = \mp 5/4; Cschu = \mp 3/4)$

22. Rewrite the following expressions in terms of exponentials , write the final result as simply as you can :

$a)\ Sinh(2.\ln x)$ $b)\ \frac{1}{Coshx - Sinhx}$

$c)\ Cosh3x - Sinh3x$ $d)\ \ln(Coshx + Sinhx) + \ln(Coshx - Sinhx)$

(ans.:$(x^4-1)/(2x^2)$; e^x ; e^{-3x} ; 0)

23. Solve the equation for x ; $tanh\ x = 3/5$. ***(ans.: ln 2)***

24. Show that the distance r from the origin O to the point $P(Coshu,Sinhu)$ on the hyperbola $x^2 - y^2 = 1$ is $r = \sqrt{Cosh2u}$.

25. If θ lies in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $Sinh\ x = tan\ \theta$. Show that :

$Cosh\ x = Sec\ \theta$, $tanh\ x = Sin\ \theta$, $Coth\ x = Csc\ \theta$, $Csch\ x = Cot\ \theta$, and $Sech\ x = Cos\ \theta$.

26. Derive the formula : $tanh^{-1}x = \frac{1}{2}\ln\frac{1+x}{1-x}$; $|x| < 1$

27. Find : $\lim_{x\to\infty}\left[Cosh^{-1}x - \ln x\right]$. ***(ans.: ln 2)***