

## Chapter three Derivatives

Let  $y = f(x)$  be a function of  $x$ . If the limit :

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

exists and is finite, we call this limit the derivative of  $f$  at  $x$  and say that  $f$  is differentiable at  $x$ .

**EX-1** – Find the derivative of the function :  $f(x) = \frac{1}{\sqrt{2x+3}}$

**Sol.:**

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{\sqrt{2(x + \Delta x) + 3}} - \frac{1}{\sqrt{2x + 3}}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{2x + 3} - \sqrt{2(x + \Delta x) + 3}}{\Delta x \cdot \sqrt{2(x + \Delta x) + 3} \sqrt{2x + 3}} \cdot \frac{\sqrt{2x + 3} + \sqrt{2(x + \Delta x) + 3}}{\sqrt{2x + 3} + \sqrt{2(x + \Delta x) + 3}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(2x + 3) - (2(x + \Delta x) + 3)}{\Delta x \cdot \sqrt{2(x + \Delta x) + 3} \sqrt{2x + 3} (\sqrt{2x + 3} + \sqrt{2(x + \Delta x) + 3})} \\ &= \frac{-2}{(2x + 3)(\sqrt{2x + 3} + \sqrt{2x + 3})} = -\frac{1}{\sqrt{(2x + 3)^3}} \end{aligned}$$

**Rules of derivatives** : Let  $c$  and  $n$  are constants,  $u$ ,  $v$  and  $w$  are differentiable functions of  $x$  :

1.  $\frac{d}{dx} c = 0$
2.  $\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx} \Rightarrow \frac{d}{dx} \left( \frac{1}{u} \right) = -\frac{1}{u^2} \frac{du}{dx}$
3.  $\frac{d}{dx} cu = c \frac{du}{dx}$
4.  $\frac{d}{dx} (u \mp v) = \frac{du}{dx} \mp \frac{dv}{dx}$  ;  $\frac{d}{dx} (u \mp v \mp w) = \frac{du}{dx} \mp \frac{dv}{dx} \mp \frac{dw}{dx}$
5.  $\frac{d}{dx} (u.v) = u \cdot \frac{dv}{dx} + v \frac{du}{dx}$

$$\text{and } \frac{d}{dx}(u.v.w) = u.v \frac{dw}{dx} + u.w \frac{dv}{dx} + v.w \frac{du}{dx}$$

$$6. \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \quad \text{where } v \neq 0$$

**EX-2-** Find  $\frac{dy}{dx}$  for the following functions :

$$a) y = (x^2 + 1)^5$$

$$b) y = [(5-x)(4-2x)]^2$$

$$c) y = (2x^3 - 3x^2 + 6x)^{-5}$$

$$d) y = \frac{12}{x} - \frac{4}{x^3} + \frac{3}{x^4}$$

$$e) y = \frac{(x^2 + x)(x^2 - x + 1)}{x^3}$$

$$f) y = \frac{x^2 - 1}{x^2 + x - 2}$$

**Sol.-**

$$a) \frac{dy}{dx} = 5(x^2 + 1)^4 \cdot 2x = 10x(x^2 + 1)^4$$

$$b) \frac{dy}{dx} = 2[(5-x)(4-2x)][-2(5-x) - (4-2x)]$$

$$= 8(5-x)(2-x)(2x-7)$$

$$c) \frac{dy}{dx} = -5(2x^3 - 3x^2 + 6x)^{-6} (6x^2 - 6x + 6)$$

$$= -30(2x^3 - 3x^2 + 6x)^{-6} (x^2 - x + 1)$$

$$d) y = 12x^{-1} - 4x^{-3} + 3x^{-4} \Rightarrow \frac{dy}{dx} = -12x^{-2} + 12x^{-4} - 12x^{-5}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{12}{x^2} + \frac{12}{x^4} - \frac{12}{x^5}$$

$$e) y = \frac{(x+1)(x^2 - x + 1)}{x^3} \Rightarrow$$

$$\frac{dy}{dx} = \frac{x^3[(x^2 - x + 1) + (x+1)(2x-1)] - 3x^2(x+1)(x^2 - x + 1)}{x^6} = -\frac{3}{x^4}$$

$$f) \frac{dy}{dx} = \frac{2x(x^2 + x - 2) - (x^2 - 1)(2x + 1)}{(x^2 + x - 2)^2} = \frac{x^2 - 2x + 1}{(x^2 + x - 2)^2}$$

The Chain Rule:

1. Suppose that  $h = g \circ f$  is the composite of the differentiable functions  $y = g(t)$  and  $x = f(t)$ , then  $h$  is a differentiable function of  $x$  whose derivative at each value of  $x$  is :

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dx}{dt}$$

2. If  $y$  is a differentiable function of  $t$  and  $t$  is differentiable function of  $x$ , then  $y$  is a differentiable function of  $x$  :

$$y = g(t) \text{ and } t = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx}$$

EX-3 – Use the chain rule to express  $dy / dx$  in terms of  $x$  and  $y$  :

a)  $y = \frac{t^2}{t^2 + 1}$       and  $t = \sqrt{2x + 1}$   
b)  $y = \frac{1}{t^2 + 1}$       and  $x = \sqrt{4t + 1}$   
c)  $y = \left(\frac{t-1}{t+1}\right)^2$       and  $x = \frac{1}{t^2} - 1$  at  $t = 2$   
d)  $y = 1 - \frac{1}{t}$       and  $t = \frac{1}{1-x}$  at  $x = 2$

Sol.-

a)  $y = \frac{t^2}{t^2 + 1} \Rightarrow \frac{dy}{dt} = \frac{2t(t^2 + 1) - 2t \cdot t^2}{(t^2 + 1)^2} = \frac{2t}{(t^2 + 1)^2}$   
 $t = (2x + 1)^{\frac{1}{2}} \Rightarrow \frac{dt}{dx} = \frac{1}{2} \cdot (2x + 1)^{-\frac{1}{2}} \cdot 2 = \frac{1}{\sqrt{2x + 1}}$   
 $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2t}{(t^2 + 1)^2} \cdot \frac{1}{\sqrt{2x + 1}} = \frac{2\sqrt{2x + 1}}{((2x + 1) + 1)^2} \cdot \frac{1}{\sqrt{2x + 1}} = \frac{1}{2(x + 1)^2}$

$$\begin{aligned}
 b) \quad y &= (t^2 + 1)^{-1} \Rightarrow \frac{dy}{dx} = -2t(t^2 + 1)^{-2} = -\frac{2t}{(t^2 + 1)^2} \\
 x &= (4t + 1)^{\frac{1}{2}} \Rightarrow \frac{dx}{dt} = \frac{1}{2}(4t + 1)^{-\frac{1}{2}} \cdot 4 = \frac{2}{\sqrt{4t + 1}} \\
 \frac{dy}{dx} &= \frac{dy}{dt} \div \frac{dx}{dt} = -\frac{2t}{(t^2 + 1)^2} \div \frac{2}{\sqrt{4t + 1}} = -\frac{t\sqrt{4t + 1}}{(t^2 + 1)^2} \\
 &= -\frac{x^2 - 1}{4} \cdot x \div \frac{1}{y^2} = -\frac{xy^2(x^2 - 1)}{4}
 \end{aligned}$$

$$\text{where } x = \sqrt{4t + 1} \Rightarrow t = \frac{x^2 - 1}{4}$$

$$\text{where } y = \frac{1}{t^2 + 1} \Rightarrow t^2 + 1 = \frac{1}{y}$$

$$\begin{aligned}
 c) \quad y &= \left(\frac{t-1}{t+1}\right)^2 \Rightarrow \frac{dy}{dt} = 2\left(\frac{t-1}{t+1}\right) \frac{t+1 - (t-1)}{(t+1)^2} = \frac{4(t-1)}{(t+1)^3} \\
 &\Rightarrow \left[\frac{dy}{dt}\right]_{t=2} = \frac{4(2-1)}{(2+1)^3} = \frac{4}{27}
 \end{aligned}$$

$$x = \frac{1}{t^2} - 1 \Rightarrow \frac{dx}{dt} = -\frac{2}{t^3} \Rightarrow \left[\frac{dx}{dt}\right]_{t=2} = -\frac{2}{2^3} = -\frac{1}{4}$$

$$\left[\frac{dy}{dx}\right]_{t=2} = \left[\frac{dy}{dt} \div \frac{dx}{dt}\right]_{t=2} = \frac{4}{27} \div \left(-\frac{1}{4}\right) = -\frac{16}{27}$$

$$d) \quad t = \frac{1}{1-x} = \frac{1}{1-2} = -1 \quad \text{at } x = 2$$

$$y = 1 - \frac{1}{t} \Rightarrow \frac{dy}{dt} = \frac{1}{t^2} \Rightarrow \left[\frac{dy}{dt}\right]_{t=-1} = \frac{1}{(-1)^2} = 1$$

$$t = (1-x)^{-1} \Rightarrow \frac{dt}{dx} = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$\Rightarrow \left[\frac{dt}{dx}\right]_{x=2} = \frac{1}{(1-2)^2} = 1$$

$$\left[\frac{dy}{dx}\right]_{x=2} = \left[\frac{dy}{dt}\right]_{x=2} \cdot \left[\frac{dt}{dx}\right]_{x=2} = 1 * 1 = 1$$

**Higher derivatives** : If a function  $y = f(x)$  possesses a derivative at every point of some interval, we may form the function  $f'(x)$  and talk

about its derivate , if it has one . The procedure is formally identical with that used before , that is :

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x}$$

if the limit exists .

This derivative is called the second derivative of  $y$  with respect to  $x$  . It is written in a number of ways , for example,

$$y'' , f''(x) , \text{ or } \frac{d^2 f(x)}{dx^2} .$$

In the same manner we may define third and higher derivatives , using similar notations . The  $n$ th derivative may be written :

$$y^{(n)} , f^{(n)}(x) , \frac{d^n y}{dx^n} .$$

**EX-4-** Find all derivatives of the following function :

$$y = 3x^3 - 4x^2 + 7x + 10$$

**Sol.-**

$$\begin{aligned} \frac{dy}{dx} &= 9x^2 - 8x + 7 & , & \quad \frac{d^2 y}{dx^2} = 18x - 8 \\ \frac{d^3 y}{dx^3} &= 18 & , & \quad \frac{d^4 y}{dx^4} = 0 = \frac{d^5 y}{dx^5} = \dots \end{aligned}$$

**Ex-5** – Find the third derivative of the following function :

$$y = \frac{1}{x} + \sqrt{x^3}$$

**Sol.-**

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{x^2} + \frac{3}{2}x^{\frac{1}{2}} \\ \frac{d^2 y}{dx^2} &= \frac{2}{x^3} + \frac{3}{4}x^{-\frac{1}{2}} \\ \frac{d^3 y}{dx^3} &= -\frac{6}{x^4} - \frac{3}{8}x^{-\frac{3}{2}} & \Rightarrow & \quad \frac{d^3 y}{dx^3} = -\frac{6}{x^4} - \frac{3}{8\sqrt{x^3}} \end{aligned}$$

**Implicit Differentiation:** If the formula for  $f$  is an algebraic combination of powers of  $x$  and  $y$ . To calculate the derivatives of these implicitly defined functions, we simply differentiate both sides of the defining equation with respect to  $x$ .

**EX-6-** Find  $\frac{dy}{dx}$  for the following functions:

$$a) x^2 \cdot y^2 = x^2 + y^2$$

$$b) (x+y)^3 + (x-y)^3 = x^4 + y^4$$

$$c) \frac{x-y}{x-2y} = 2 \text{ at } P(3,1)$$

$$d) xy + 2x - 5y = 2 \text{ at } P(3,2)$$

**Sol.**

$$a) x^2 \left( 2y \frac{dy}{dx} \right) + y^2 (2x) = 2x + 2y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2 y - y}$$

$$b) 3(x+y)^2 \left( 1 + \frac{dy}{dx} \right) + 3(x-y)^2 \left( 1 - \frac{dy}{dx} \right) = 4x^3 + 4y^3 \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 3(x+y)^2 - 3(x-y)^2}{3(x+y)^2 - 3(x-y)^2 - 4y^3} \Rightarrow \frac{dy}{dx} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

$$c) \frac{(x-2y) \left( 1 - \frac{dy}{dx} \right) - (x-y) \left( 1 - 2 \frac{dy}{dx} \right)}{(x-2y)^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \left[ \frac{dy}{dx} \right]_{(3,1)} = \frac{1}{3}$$

$$d) x \frac{dy}{dx} + y + 2 - 5 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{5-x} \Rightarrow \left[ \frac{dy}{dx} \right]_{(3,2)} = \frac{2+2}{5-3} = 2$$

**Exponential functions:** If  $u$  is any differentiable function of  $x$ , then :

$$7) \frac{d}{dx} a^u = a^u \cdot \ln a \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}$$

**EX-7** – Find  $\frac{dy}{dx}$  for the following functions :

a)  $y = 2^{3x}$

b)  $y = 2^x \cdot 3^x$

c)  $y = (2^x)^2$

d)  $y = x \cdot 2^{x^2}$

e)  $y = e^{(x+e^{5x})}$

f)  $y = e^{\sqrt{1+5x^2}}$

**Sol.-**

a)  $y = 2^{3x} \Rightarrow \frac{dy}{dx} = 2^{3x} * 3 \ln 2$

b)  $y = 2^x \cdot 3^x \Rightarrow y = 6^x \Rightarrow \frac{dy}{dx} = 6^x \cdot \ln 6$

c)  $y = (2^x)^2 \Rightarrow y = 2^{2x} \Rightarrow \frac{dy}{dx} = 2^{2x} \ln 2 \cdot 2 = 2^{2x+1} \ln 2$

d)  $y = x \cdot 2^{x^2} \Rightarrow \frac{dy}{dx} = x \cdot 2^{x^2} \ln 2 \cdot 2x + 2^{x^2} = 2^{x^2} (2x^2 \ln 2 + 1)$

e)  $y = e^{(x+e^{5x})} \Rightarrow \frac{dy}{dx} = e^{(x+e^{5x})} (1 + 5e^{5x})$

f)  $y = e^{(1+5x^2)^{\frac{1}{2}}} \Rightarrow \frac{dy}{dx} = e^{(1+5x^2)^{\frac{1}{2}}} \cdot \frac{1}{2} (1+5x^2)^{-\frac{1}{2}} \cdot 10x = e^{\sqrt{1+5x^2}} \frac{5x}{\sqrt{1+5x^2}}$

**Logarithm functions** : If  $u$  is any differentiable function of  $x$ , then :

8)  $\frac{d}{dx} \log_a u = \frac{1}{u \cdot \ln a} \cdot \frac{du}{dx}$  and  $\frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}$

**EX-8** – Find  $\frac{dy}{dx}$  for the following functions :

a)  $y = \log_{10} e^x$

b)  $y = \log_5 (x+1)^2$

c)  $y = \log_2 (3x^2 + 1)^3$

d)  $y = \left[ \ln(x^2 + 2)^2 \right]^3$

e)  $y + \ln(xy) = 1$

f)  $y = \frac{(2x^3 - 4)^{\frac{2}{3}} \cdot (2x^2 + 3)^{\frac{5}{2}}}{(7x^3 + 4x - 3)^2}$

**Sol. –**

$$\begin{aligned}
a) \quad y = \log_{10} e^x &\Rightarrow y = x \log_{10} e \Rightarrow \frac{dy}{dx} = \log_{10} e = \frac{\ln e}{\ln 10} = \frac{1}{\ln 10} \\
b) \quad y = \log_5 (x+1)^2 &= 2 \log_5 (x+1) \Rightarrow \frac{dy}{dx} = \frac{2}{(x+1) \ln 5} \\
c) \quad y = 3 \log_2 (3x^2 + 1) &\Rightarrow \frac{dy}{dx} = \frac{3}{3x^2 + 1} \cdot \frac{6x}{\ln 2} = \frac{18x}{(3x^2 + 1) \ln 2} \\
d) \quad \frac{dy}{dx} &= 3 \left[ 2 \ln(x^2 + 2) \right]^2 \cdot \frac{2}{x^2 + 2} \cdot 2x = \frac{48x \left[ \ln(x^2 + 2) \right]^2}{x^2 + 2} \\
e) \quad y + \ln x + \ln y &= 1 \Rightarrow \frac{dy}{dx} + \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x(y+1)} \\
f) \quad \ln y &= \frac{2}{3} \ln(2x^3 - 4) + \frac{5}{2} \ln(2x^2 + 3) - 2 \ln(7x^3 + 4x - 3) \\
&\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{3} \cdot \frac{6x^2}{2x^3 - 4} + \frac{5}{2} \cdot \frac{4x}{2x^2 + 3} - 2 \cdot \frac{21x^2 + 4}{7x^3 + 4x - 3} \\
&\Rightarrow \frac{dy}{dx} = 2y \left[ \frac{2x^2}{2x^3 - 4} + \frac{5x}{2x^2 + 3} - \frac{21x^2 + 4}{7x^3 + 4x - 3} \right]
\end{aligned}$$

**Trigonometric functions** : If  $u$  is any differentiable function of  $x$ , then :

$$\begin{aligned}
9) \quad \frac{d}{dx} \sin u &= \cos u \cdot \frac{du}{dx} \\
10) \quad \frac{d}{dx} \cos u &= -\sin u \cdot \frac{du}{dx} \\
11) \quad \frac{d}{dx} \tan u &= \sec^2 u \cdot \frac{du}{dx} \\
12) \quad \frac{d}{dx} \cot u &= -\csc^2 u \cdot \frac{du}{dx} \\
13) \quad \frac{d}{dx} \sec u &= \sec u \cdot \tan u \cdot \frac{du}{dx} \\
14) \quad \frac{d}{dx} \csc u &= -\csc u \cdot \cot u \cdot \frac{du}{dx}
\end{aligned}$$

**EX-9**- Find  $\frac{dy}{dx}$  for the following functions :



$$a) y = \tan(3x^2)$$

$$b) y = (\csc x + \cot x)^2$$

$$c) y = 2\sin \frac{x}{2} - x \cos \frac{x}{2}$$

$$d) y = \tan^2(\cos x)$$

$$e) x + \tan(xy) = 0$$

$$f) y = \sec^4 x - \tan^4 x$$

**Sol.-**

$$a) \frac{dy}{dx} = \sec^2(3x^2) \cdot 6x = 6x \cdot \sec^2(3x^2)$$

$$b) \frac{dy}{dx} = 2(\csc x + \cot x)(-\csc x \cdot \cot x - \csc^2 x) = -2 \csc x \cdot (\csc x + \cot x)^2$$

$$c) \frac{dy}{dx} = 2 \cos \frac{x}{2} \cdot \frac{1}{2} - \left[ x(-\sin \frac{x}{2}) \cdot \frac{1}{2} + \cos \frac{x}{2} \right] = \frac{x}{2} \cdot \sin \frac{x}{2}$$

$$d) \frac{dy}{dx} = 2 \tan(\cos x) \cdot \sec^2(\cos x) \cdot (-\sin x) = -2 \sin x \cdot \tan(\cos x) \cdot \sec^2(\cos x)$$

$$e) 1 + \sec^2(xy) \cdot (x \frac{dy}{dx} + y) = 0 \Rightarrow \frac{dy}{dx} = -\frac{1 + y \cdot \sec^2(xy)}{x \cdot \sec^2(xy)} = -\frac{\cos^2(xy) + y}{x}$$

$$f) \frac{dy}{dx} = 4 \sec^3 x \cdot \sec x \cdot \tan x - 4 \tan^3 x \cdot \sec^2 x = 4 \tan x \cdot \sec^2 x$$

**EX-10- Prove that :**

$$a) \frac{d}{dx} \tan u = \sec^2 u \cdot \frac{du}{dx}$$

$$b) \frac{d}{dx} \sec u = \sec u \cdot \tan u \cdot \frac{du}{dx}$$

**Proof:**

$$\begin{aligned} a) \text{ L.H.S.} &= \frac{d}{dx} \tan u = \frac{d}{dx} \frac{\sin u}{\cos u} = \frac{\cos u \cdot \cos u \cdot \frac{du}{dx} - \sin u \cdot (-\sin u) \frac{du}{dx}}{\cos^2 u} \\ &= \frac{\cos^2 u + \sin^2 u}{\cos^2 u} \cdot \frac{du}{dx} = \frac{1}{\cos^2 u} \cdot \frac{du}{dx} = \sec^2 u \cdot \frac{du}{dx} = \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} b) \text{ L.H.S.} &= \frac{d}{dx} \sec u = \frac{d}{dx} \frac{1}{\cos u} = -\frac{1}{\cos^2 u} (-\sin u) \frac{du}{dx} \\ &= \frac{1}{\cos u} \cdot \frac{\sin u}{\cos u} \cdot \frac{du}{dx} = \sec u \cdot \tan u \cdot \frac{du}{dx} = \text{R.H.S.} \end{aligned}$$

**The inverse trigonometric functions :** If  $u$  is any differentiable function