Maxwell's equations

The mathematical physical principles to rule the electromagnetic problems are the Maxwell equations. James Clerk Maxwell (1831-1879) elegantly integrated the electric, magnetic, and the electro-magnetic induction theories prior to his era and formed a set of differential equations. This integration has been known as the Maxwell equations thereafter. Maxwell's Equations are four of the most influential equations in science: Gauss's law for electric fields, Gauss's law for magnetic fields, Faraday's law, and the Ampere–Maxwell law.

In this lecture, plain-language explanations of the physical meaning of each symbol in the equation, for both the integral and differential forms is presented.

The Constitutive relations

Similar to the constitutive relations in continuous medium mechanics, there are also constitutive relationships in electromagnetics. Constitutive relations describe the medium's properties and effects when two physical quantities are related. It can be viewed as the description of response of the medium as a system to certain input. In electromagnetics, there are four fundamental constitutive relationships to describe the response of a medium to a variety of electromagnetic input. Two of them describe the relationship between the electric field **E** and the conductive current **J**, and the electric displacement **D**, and the other two describe the relationship between the magnetic field **H** and the magnetic induction **B**, and the magnetic polarization **M**. Quantitatively, these four constitutive relationships are

$$\boxed{\begin{array}{l} J = \sigma . E \\ D = \varepsilon . E \end{array}} \qquad \boxed{\begin{array}{l} M = \chi . H \\ B = \mu . H \end{array}}$$

where σ is the electric conductivity, ε the dielectric permittivity, μ the magnetic permeability, and χ the magnetic susceptibility.

EQUATION OF CONTINUITY FOR TIME VARYING FIELDS

Equation of continuity in point form is

$$\nabla . J = \frac{-\partial \rho_v}{\partial t}$$

where,

J = conduction current density (A/m^2)

ρ_v = volume charge density (C/m^2)

∇ = vector differential operator (1/m)

$$\nabla = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

Proof: Consider a closed surface enclosing a charge Q. There exists an outward flow of current given by

$$I = \oint J . ds$$

This is equation of continuity in integral form.

From the principle of conservation of charge, we have

$$I = \oint J . ds = \frac{-\partial Q}{\partial t}$$

From the divergence theorem, we have

$$I = \oint_s J . ds = \oint_v (\nabla . J) . dv$$

Thus, $I = \oint_v (\nabla . J) . dv = \frac{-\partial Q}{\partial t}$

By definition,

$$Q = \oint_v \rho_v . dv$$

where, ρ_v = volume charge density (C/m^3))

So, $$\oint_v (\nabla . J) . dv = \frac{-\partial Q}{\partial t} = \frac{-\oint_v \rho_v . dv}{\partial t}$$

The volume integrals are equal only if their integrands are equal.

Thus

$$\boxed{\nabla . J = \frac{-\partial \rho_v}{\partial t}}$$

MAXWELL'S EQUATIONS FOR TIME VARYING FIELDS

These are basically four in number Maxwell's equations in differential form and in integral form are shown in the following table (1)

Differential (or Point) Form	Integral Form	Remarks
$\nabla . D = \rho_v$	$\oint_S D \cdot dS = \int_v \rho_v dv$	Gauss's law
$\nabla . B = 0$	$\oint_S B \cdot dS = 0$	Nonexistence of magnetic monopole
$\nabla \times E = -\frac{\partial B}{\partial t}$	$\oint_L E \cdot dl = -\frac{\partial}{\partial t} \int_s B \cdot dS$	Faraday's Law
$\nabla \times H = J + \frac{\partial D}{\partial t}$	$\oint_L H \cdot dl = \int_s J \cdot dS$	Ampere's circuit law

Here,

H = magnetic field strength (A/m)

D = electric flux density, (C/m^2)

($\frac{\partial D}{\partial t}$) = displacement electric current density (A/m^2)

J = conduction current density (A/m^2)

E = electric field (V/m)

B = magnetic flux density wb/m^2 or Tesla

($\frac{\partial B}{\partial t}$) = time-derivative of magnetic flux density (wb/m^2 -sec) is called magnetic current density (V/m^2) or Tesla/sec

ρ_v = volume charge density (C/m^3)

1. Gauss's law for electric fields

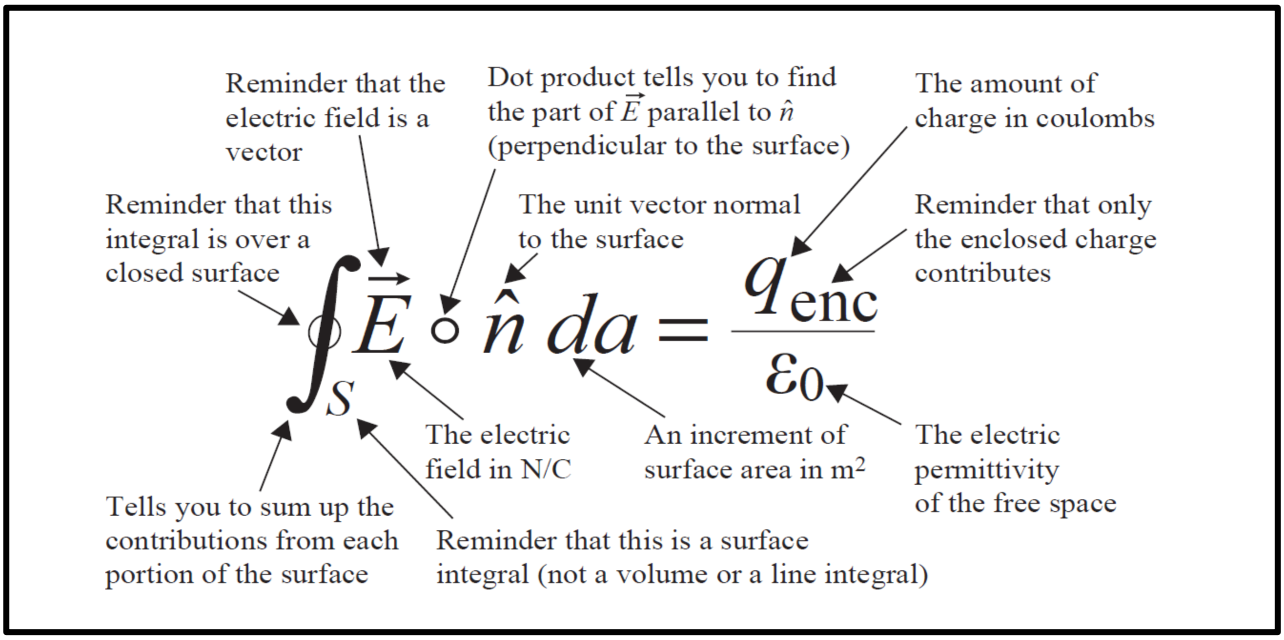
In Maxwell's Equations, you'll encounter two kinds of electric field: the electrostatic field produced by electric charge and the induced electric field produced by a changing magnetic field. Gauss's law for electric fields deals with the electrostatic field, and you'll find this law to be a powerful tool because it relates the spatial behavior of the electrostatic field to the charge distribution that produces it.

1.1 The integral form of Gauss's law

2 the main idea of Gauss's law in integral form:

Electric charge produces an electric field, and the flux of that field passing through any closed surface is proportional to the total charge contained within that surface.

the integral form is generally written like this



The left side of this equation is no more than a mathematical description of the electric flux – the number of electric field lines – passing through a closed surface S, whereas the right side is the total amount of charge contained within that surface divided by a constant called the permittivity of free space.

We know that the electric flux density $D = \varepsilon_0 . E$, then the above equation can be also written as

$$\oiint D.ds = \iiint \rho_v . dv = Q$$

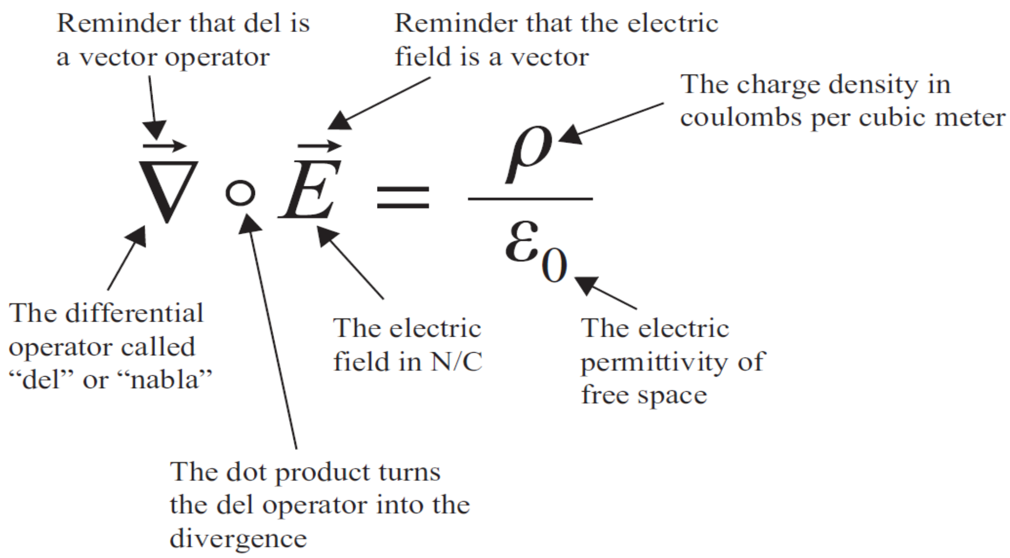
1.2 The differential form of Gauss's law

The integral form of Gauss's law for electric fields relates the electric flux over a surface to the charge enclosed by that surface – but like all of Maxwell's Equations, Gauss's law may also be cast in differential form.

the main idea of Gauss's law in differential form:

The electric field produced by electric charge diverges from positive charge and converges upon negative charge.

The differential form is generally written as



The left side of this equation is a mathematical description of the divergence of the electric field – the tendency of the field to ''flow'' away from a specified location – and the right side is the electric charge density divided by the permittivity of free space.

Where, $D = \varepsilon . E$

Thus $\boxed{\nabla . D = \rho_v}$

Proof:

From Guess law in electric field

$$\oint D.ds = Q = \oint_v \rho_v . dv$$

Applying divergence theorem; we get:

$$\oint_s D.ds = \oint_v (\nabla . D). dv$$

Thus, $\oint_v (\nabla . D). dv = \oint_v \rho_v . dv$

The volume integrals are equal only if their integrands are equal.

Thus

$$\boxed{\nabla . D = \rho_v}$$

2. Gauss's law for magnetic fields

Gauss's law for magnetic fields is similar in form but different in content from Gauss's law for electric fields. For both electric and magnetic fields, the integral form of Gauss's law involves the flux of the field over a closed surface, and the differential form specifies the divergence of the field at a point.

The key difference in the electric field and magnetic field versions of Gauss's law arises because opposite electric charges (called ''positive'' and ''negative'') may be isolated from one another, while opposite magnetic poles (called ‘‘north’’ and ‘‘south’’) always occur in pairs. As you might expect, the apparent lack of isolated magnetic poles in nature has a profound impact on the behavior of magnetic flux and on the divergence of the magnetic field.

2.1 The integral form of Gauss’s law

the main idea of Gauss’s law for magnetic fields:

The total magnetic flux passing through any closed surface is zero.

the integral form of Gauss’s law is generally written as follows:

Reminder that the magnetic field is a vector

Dot product tells you to find the part of $\vec{B}$ parallel to $\hat{n}$ (perpendicular to the surface)

The unit vector normal to the surface

$$\oint_S \vec{B} \circ \hat{n}\, da = 0$$

Reminder that this integral is over a closed surface

The magnetic field in Teslas

An increment of surface area in m2

Reminder that this is a surface integral (not a volume or a line integral)

Tells you to sum up the contributions from each portion of the surface

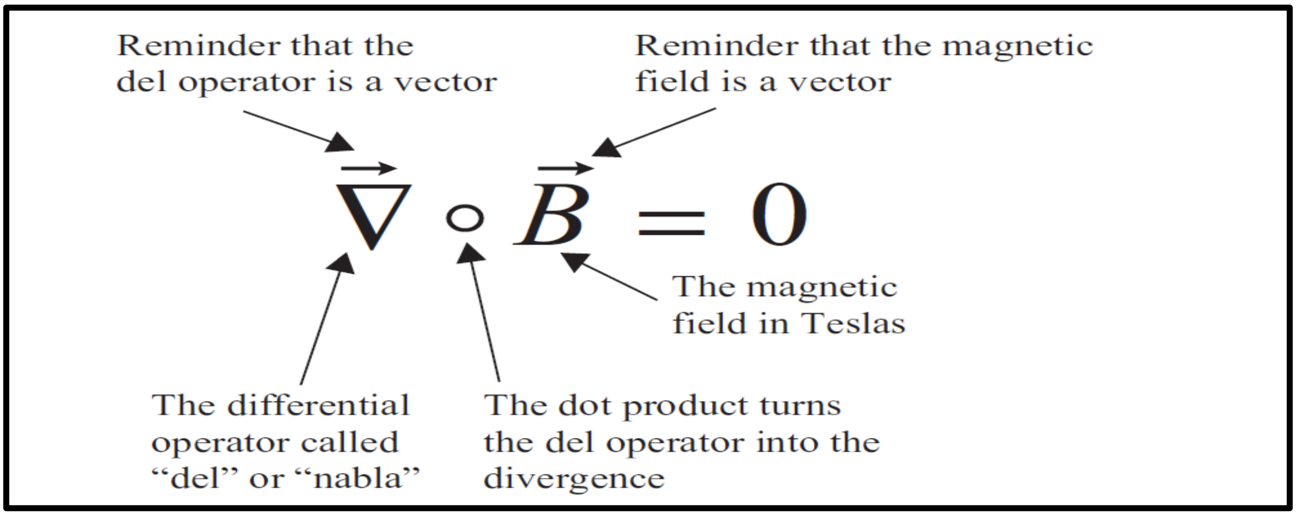
the left side of this equation is a mathematical description of the flux of a vector field through a closed surface. In this case, Gauss’s law refers to magnetic flux – the number of magnetic field lines – passing through a closed surface S. The right side is identically zero.

2.2 The differential form of Gauss’s law

the main idea of Gauss’s law:

The divergence of the magnetic field at any point is zero.

The differential form is written as



The left side of this equation is a mathematical description of the divergence of the magnetic field – the tendency of the magnetic field to ''flow'' more strongly away from a point than toward it – while the right side is simply zero. the main idea of Gauss's law in differential form:

Proof:

We have Gauss's law for magnetic fields as

$$\oint_s B.ds = 0$$

Right hand side is zero as there are no isolated magnetic charges and the magnetic flux lines are closed loops.

Applying divergence theorem to left hand side, we get

$$\oint_s B.ds = \oint_v \nabla.B.dv = 0$$

or,

$$\nabla . B = 0$$

Hence proved.

3. Faraday's law

In a series of epoch-making experiments in 1831, Michael Faraday demonstrated that an electric current may be induced in a circuit by changing the magnetic flux enclosed by the circuit. That discovery is made even more useful when extended to the general statement that a changing magnetic field produces an electric field. Such ''induced'' electric fields are very different from the fields produced by electric charge, and Faraday's law of induction is the key to understanding their behavior.

3.1 The integral form of Faraday's law

the main idea of Faraday's law:

Changing magnetic flux through a surface induces an emf in any boundary path of that surface, and a changing magnetic field induces a circulating electric field.

In other words, if the magnetic flux through a surface changes, an electric field is induced along the boundary of that surface. If a conducting material is present along that boundary, the induced electric field provides an emf that drives a current through the material. Thus quickly poking a bar magnet through a loop of wire generates an electric field within that wire, but holding the magnet in a fixed position with respect to the loop induces no electric field.

Here's an expanded view of the standard form of Faraday's law:

Reminder that the electric field is a vector

Dot product tells you to find the part of $\vec{E}$ parallel to $d\vec{l}$ (along path C)

The magnetic flux through any surface bounded by C

An incremental segment of path C

$$\oint_C \vec{E} \circ d\vec{l} = -\frac{d}{dt}\overbrace{\int_S \vec{B} \circ \hat{n}\, da}$$

The electric field in V/m

The rate of change with time

Tells you to sum up the contributions from each portion of the closed path C in a direction given by the right-hand rule

Reminder that this is a line integral (not a surface or a volume integral)

3.2 The differential form of Faraday's law

the main idea of Faraday's law in differential form:

A circulating electric field is produced by a magnetic field that changes with time.

The differential form of Faraday's law is generally written as

Reminder that the del operator is a vector

Reminder that the electric field is a vector

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The rate of change of the magnetic field with time

The differential operator called "del" or "nabla"

The electric field in V/m

The cross-product turns the del operator into the curl

The left side of this equation is a mathematical description of the curl of the electric field – the tendency of the field lines to circulate around a point. The right side represents the rate of change of the magnetic field over time.

Proof:

According to Faraday's law, $emf = -\frac{\partial \psi}{\partial t}$

ψ = magnetic flux, (wb)

and by definition,

$$emf = \oint_l E.dl$$

$$\oint_l E.dl = -\frac{\partial \psi}{\partial t}$$

But,

$$\psi = \oint_s B.ds$$

$$\oint_l E.dl = -\frac{\partial \oint_s B.ds}{\partial t}$$

Applying Stokes' theorem to left hand side, we get

$$\oint_l E.dl = \oint_s (\nabla \times E).ds$$

$$\oint_s (\nabla \times E).ds = -\frac{\partial \oint_s B.ds}{\partial t}$$

Two surface integrals are equal only if their integrands are equal,
that is,

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{(proved)}$$

4. The Ampere–Maxwell law

''Ampere's law'' relating a steady electric current to a circulating magnetic field was well known by the time James Clerk Maxwell began his work in the field in the 1850s. However, Ampere's law was known to apply only to static situations involving steady currents. It was Maxwell's addition of another source term – a changing electric flux – that extended the applicability of Ampere's law to time-dependent conditions. More importantly, it was the presence of this term in the equation now called the Ampere–Maxwell law that allowed Maxwell to discern the electromagnetic nature of light and to develop a comprehensive theory of electromagnetism.

4.1 The integral form of the Ampere–Maxwell law

the main idea of the Ampere–Maxwell law:

> An electric current or a changing electric flux through a surface produces a circulating magnetic field around any path that bounds that surface.

The integral form of the Ampere–Maxwell law is generally written as

$$\oint_C \vec{B} \circ d\vec{l} = \mu_0 \left(I_{enc} + \varepsilon_0 \frac{d}{dt} \underbrace{\int_S \vec{E} \circ \hat{n}\, da}_{\text{The electric flux through a surface bounded by } C} \right)$$

Reminder that the magnetic field is a vector

Dot product tells you to find the part of $\vec{B}$ parallel to $d\vec{l}$ (along path C)

An incremental segment of path C

The electric current in amperes

The rate of change with time

The magnetic field in teslas

Reminder that only the enclosed current contributes

Tells you to sum up the contributions from each portion of closed path C in a direction given by the right-hand rule

The magnetic permeability of free space

The electric permittivity of free space

The left side of this equation is a mathematical description of the circulation of the magnetic field around a closed path C. The right side includes two sources for the magnetic field; a steady conduction current and a changing electric flux through any surface S bounded by path C.

4.2 The differential form of the Ampere–Maxwell law

the main idea of the differential form of the Ampere–Maxwell law:

A circulating magnetic field is produced by an electric current and by an electric field that changes with time.

The differential form of the Ampere–Maxwell law is generally written as

Reminder that the del operator is a vector

Reminder that the magnetic field is a vector

Reminder that the current density is a vector

The rate of change of the electric field with time

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

The differential operator called “del” or “nabla”

The magnetic field in teslas

The magnetic permeability of free space

The electric current density in amperes per square meter

The electric permittivity of free space

The cross-product turns the del operator into the curl

The left side of this equation is a mathematical description of the curl of the magnetic field – the tendency of the field lines to circulate around a point. The two terms on the right side represent the electric current density and the time rate of change of the electric field.

Proof:

From Ampere's circuital law, we have

$$\nabla \times H = J$$

Take dot product on both sides

$$\nabla . \nabla \times H = \nabla . J$$

As the divergence of curl of a vector is zero, $(\nabla . \nabla \times H = 0)$

Right side hand $\nabla . J = 0$

But the equation of continuity in point form is

$$\nabla . J = \frac{-\partial \rho_v}{\partial t}$$

This means that if $\nabla \times H = J$ is true, it is resulting in $\nabla . J = 0$.

As the equation of continuity is more fundamental, Ampere's circuital law should be modified. Hence we can write

$$\nabla \times H = J + F$$

Take dot product on both sides

$$\nabla . \nabla \times H = \nabla . J + \nabla . F$$

that is, ∇.∇×H=0=∇.J+∇.F

Substituting the value of □.J from the equation of continuity in the above expression, we get

$$\nabla . J + \nabla . F = 0$$

$$\nabla . J = -\nabla . F$$

Or $$\nabla . F = \frac{\partial \rho_v}{\partial t}$$

The point form of Gauss's law is

$$\nabla . D = \rho_v$$

From the above expressions, we get

$$\nabla . F = \frac{\partial}{\partial t}(\nabla . D)$$

The divergence of two vectors are equal only if the vectors are identical,

that is, F = $= \frac{\partial D}{\partial t}$

So,

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

Hence proved.

TIME-HARMONIC ELECTROMAGNETIC FIELDS

Maxwell's equations in differential and integral forms, for general time-varying electromagnetic fields, were presented in Sections above. However, in many practical systems involving electromagnetic waves, the time variations are of cosinusoidal form and are referred to as time-harmonic. In general, such time variations can be represented by $e^{j\omega t}$, and the instantaneous electromagnetic field vectors can be related to their complex forms in a very simple manner. Thus for time-harmonic fields, we can relate the instantaneous fields, current density and charge (represented by script letters) to their complex forms (represented by roman letters) by

$$\mathcal{E}(x,y,z,t) = Re[E(x,y,z)e^{j\omega t}]$$

$$\mathcal{H}(x,y,z,t) = Re[H(x,y,z)e^{j\omega t}]$$

$$\mathcal{D}(x,y,z,t) = Re\left[D(x,y,z)e^{j\omega t}\right]$$

$$\mathcal{B}(x,y,z,t) = Re[B(x,y,z)e^{j\omega t}]$$

$$\mathcal{J}(x,y,z,t) = Re\left[J(x,y,z)e^{j\omega t}\right]$$

$$\mathcal{q}(x,y,z,t) = Re[q(x,y,z)e^{j\omega t}]$$

where $\mathcal{E}, \mathcal{H}, \mathcal{D}, \mathcal{B}, \mathcal{J},$ and $\mathcal{q}$ represent the instantaneous field vectors, current density and charge, while E, H, D, B, J, and q represent the corresponding complex spatial forms which are only a function of position. In this book we have chosen to represent the instantaneous quantities by the real part of the product of the corresponding complex spatial quantities with $e^{j\omega t}$.

Maxwell's Equations in Differential and Integral Forms

Maxwell's equations and the continuity equation in differential form for time harmonic fields can be written in terms of the complex field vectors as shown in Table (2).

By examining the two forms in Table (1), we see that one form can be obtained from the other by doing the following:

1. Replace the instantaneous field vectors by the corresponding complex spatial forms, or vice versa.

2. Replace $\partial/\partial t$ by $j\omega(\partial/\partial t = j\omega)$, or vice versa.

Time varying		Time harmonic	
Integration form	Differential form	Integration form	Differential form
$\oint_s \mathcal{J}.ds = \frac{-\partial\rho_v}{\partial t}$	$\nabla.\mathcal{J} = \frac{-\partial\rho_v}{\partial t}$	$\oint_s J.ds = -j\omega\rho_v$	$\nabla.J = -j\omega\rho_v$
$\oiint \mathcal{D}.ds = Q$	$\mathcal{D} = \rho_v$	$\oiint D.ds = Q$	$D = \rho_v$
$\oint_s \mathcal{B}.ds = 0$	$\nabla.\mathcal{B} = 0$	$\oint_s B.ds = 0$	$\nabla.B = 0$
$\oint_l \mathcal{E}.dl = -\frac{\partial}{\partial t}\oint_s \mathcal{B}.ds$	$\nabla\times\mathcal{E} = -\frac{\partial\mathcal{B}}{\partial t}$	$\oint_l E.dl = -j\omega\oint_s B.ds$	$\nabla\times E = -j\omega\frac{\partial B}{\partial t}$
$\oint_l^- \mathcal{H}.dl = \mathcal{J} + \frac{\partial}{\partial t}\oint_s^- \mathcal{D}.ds$	$\nabla\times\mathcal{H} = \mathcal{J} + \frac{\partial\mathcal{D}}{\partial t}$	$\oint_l^- H.dl = J + j\omega\oint_s^- D.ds$	$\nabla\times H = J + j\omega\mathrm{D}$