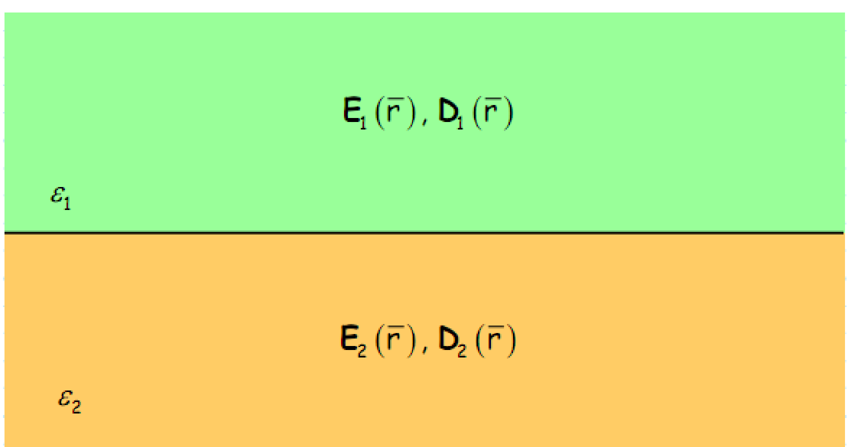
Dielectric Boundary Conditions

So far, we have considered the existence of the electric field in a homogeneous medium. If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called boundary conditions. These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known. Obviously, the conditions will be dictated by the types of material the media are made of. We shall consider the boundary conditions at an interface separating

1- **Dielectric-Dielectric Interface**
2- **Conductive-Dielectric Interface**
3- **Conductive-free space Interface**

1-Dielectric-Dielectric Interface

Consider the **interface** between two dissimilar **dielectric** regions:



Say that an electric field is present in both regions, thus producing also an electric flux density $D(\vec{r}) = \varepsilon E(\vec{r})$.

Q: How are the fields in dielectric **region 1** (i.e., $E_1(\vec{r}), D_1(\vec{r})$ related to the fields in **region 2** (i.e., $E_2(\vec{r}), D_2(\vec{r})$)

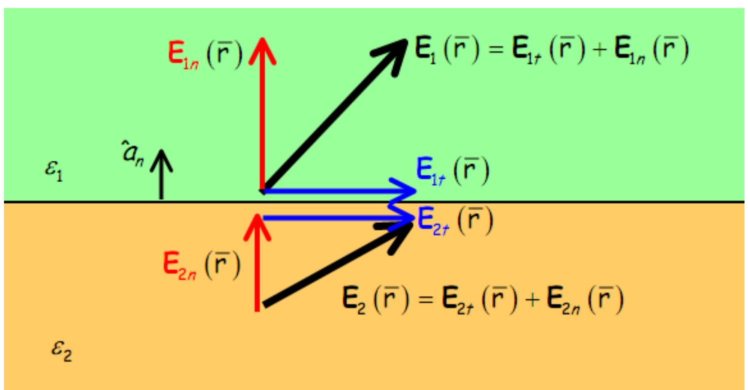
Answer: They must satisfy the dielectric **boundary conditions**!

First, let's write the fields at the dielectric interface in terms of their **normal** ($E_n(\vec{r})$) and **tangential** ($E_t(\vec{r})$) vector components:

The electric field in the two media can be expressed as

Region 1: $\vec{E}_1(r) = \vec{E}_{1t}(r) + \vec{E}_{1n}(r)$

Region 2: $\vec{E}_2(r) = \vec{E}_{2t}(r) + \vec{E}_{2n}(r)$



Our first boundary condition states that the tangential component of the electric field is continuous across a boundary. In other words:

$$\boxed{\mathbf{E}_{1t}(\bar{r}_b) = \mathbf{E}_{2t}(\bar{r}_b)}$$

Where $\vec{r_b}$ denotes any point on the boundary (e.g., dielectric interface).

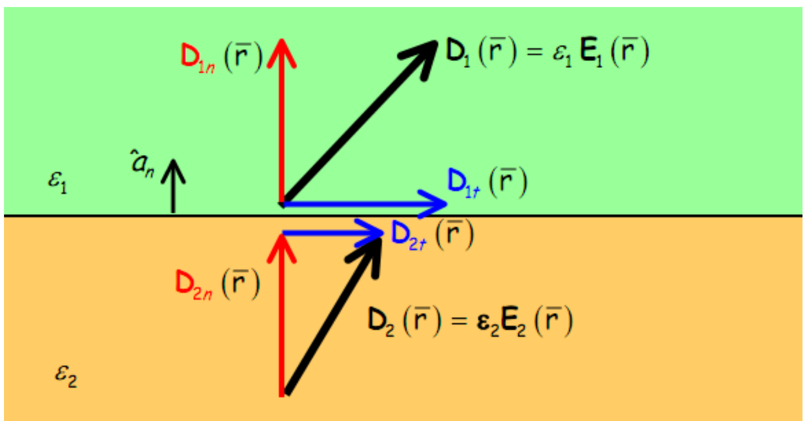
(**The tangential component of the electric field at one side of the dielectric boundary is equal to the tangential component at the other side**).

Since$D(\vec{r}) = \varepsilon E(\vec{r})$, these boundary conditions can likewise be expressed as:

$$\boxed{\begin{gathered} E_{1t}(\bar{r}_b) = E_{2t}(\bar{r}_b) \\ \frac{D_{1t}(\bar{r}_b)}{\varepsilon_1} = \frac{D_{2t}(\bar{r}_b)}{\varepsilon_2} \end{gathered}}$$

The tangential components of $D(\vec{r})$ under goes some change across the boundary. So, $D(\vec{r})$ is said to be discontinouse across the boundary.

We can likewise consider the electric flux densities on the dielectric interface in terms of their normal and tangential components:



The second dielectric boundary condition states that the normal vector component of the electric flux density is continuous across the dielectric boundary. In other words:

$$\boxed{D_{1n}(\vec{r}) - D_{2n}(\vec{r}) = \rho_v}$$

If there is no charge exists in the boundary, $(\rho_v = 0)$, then

$$\boxed{D_{1n}(\bar{r}_b) = D_{2n}(\bar{r}_b)}$$

Where $\vec{r_b}$ denotes any point on the dielectric boundary (i.e., dielectric interface).

<u>The normal components of electric flux density $D(\vec{r})$ at one side of the dielectric boundary is equal to the normal components at the other side</u>

$D_n(\vec{r})$ undergoes no change on the boundary and it is continous across the boundary

$$\mathbf{D}_{1n}(\bar{r}_b) = \mathbf{D}_{2n}(\bar{r}_b)$$

$$\varepsilon_1 \mathbf{E}_{1n}(\bar{r}_b) = \varepsilon_2 \mathbf{E}_{2n}(\bar{r}_b)$$

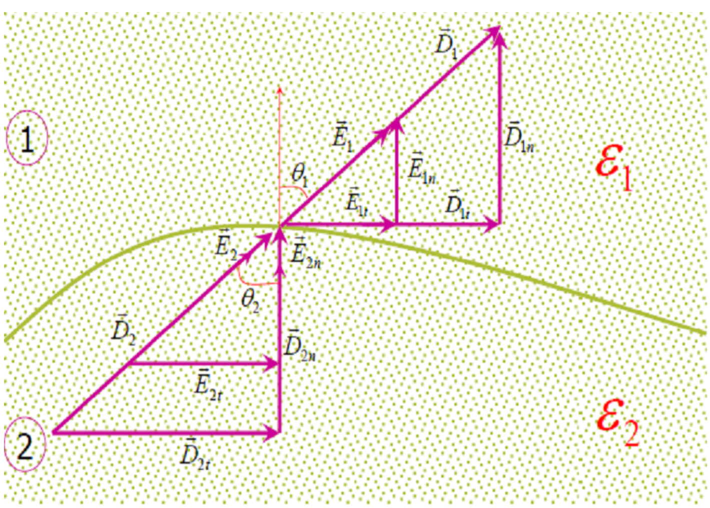
The normal components of $E(\vec{r})$ under goes some change across the boundary. So, $E(\vec{r})$ is said to be discontinouse across the boundary.

We can likewise consider the electric flux densities on the dielectric interface in terms of their normal and tangential components:

<u>MAKE SURE YOU UNDERSTAND THIS:</u>

These boundary conditions describe the relationships of the vector fields at the dielectric interface only (i.e., at points $\vec{r} = \vec{r}_b$).They say nothing about the value of the fields at points above or below the interface.

Reflection law



$$E_{1t} = E_{2t} \qquad\qquad D_{1n} = D_{2n}$$

$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \text{ -- } (1) \qquad D_1 \cos\theta_1 = D_2 \cos\theta_2$$

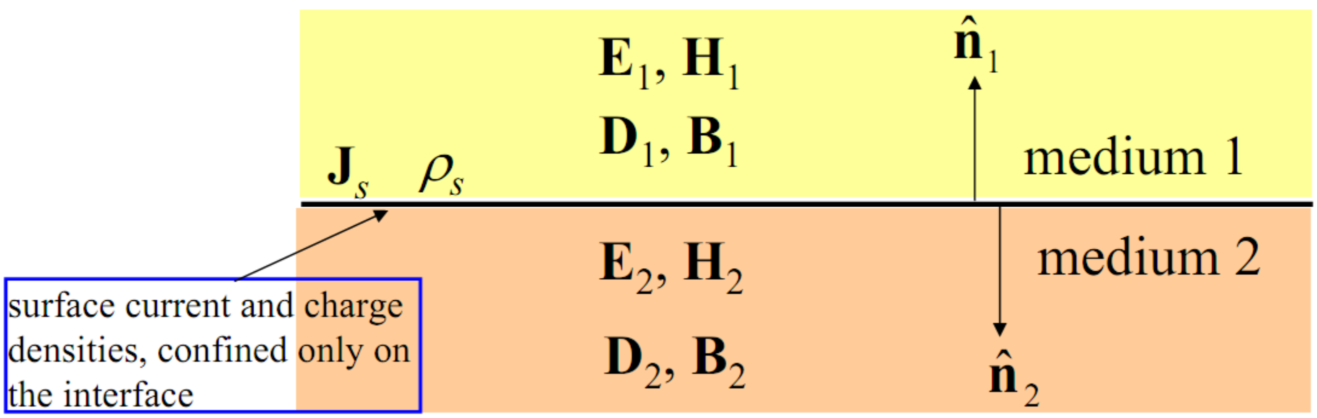
$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2 \text{ --- } (2)$$

$$(1)/(2) \Rightarrow \frac{E_1 \sin\theta_1}{\varepsilon_1 E_1 \cos\theta_1} = \frac{E_2 \sin\theta_2}{\varepsilon_2 E_2 \cos\theta_2}$$

$$\varepsilon_2 \tan\theta_1 = \varepsilon_1 \tan\theta_2$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_0\varepsilon_{r1}}{\varepsilon_0\varepsilon_{r2}} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \qquad \boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}}$$

Boundary conditions of magnetic field $H(\vec{r})$:



We get the condition on the tangential component of the H field as

$$H_{1t}(\vec{r}) - H_{2t}(\vec{r}) = J_s$$

If J_s=0

$$H_{1t}(\vec{r}) - H_{2t}(\vec{r}) = 0$$

Then

$$\boxed{\boldsymbol{H_{1t}(\vec{r}) = H_{2t}(\vec{r})}}$$

The normal component of magnetic flux density

$$B_{1n}(\vec{r}) - B_{2n}(\vec{r}) = 0$$

Then

$$\boxed{\boldsymbol{B_{1n}(\vec{r}) = B_{2n}(\vec{r})}}$$

Which says the normal component of the magnetic field is always continuous.

In words, this means:

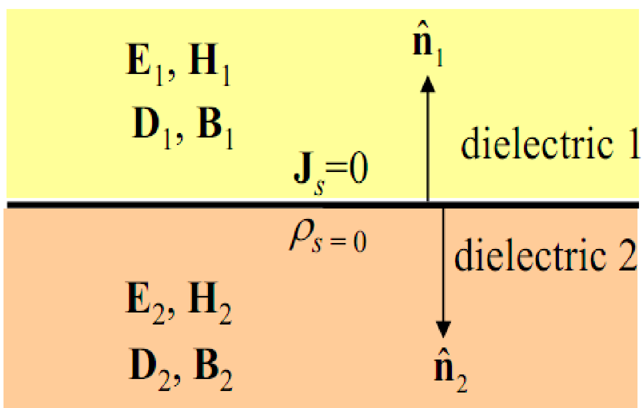
1- The tangential electric field is continuous.

2- The tangential magnetic field is discontinuous if $J_S = 0$.

3-The normal component of the magnetic flux is continuous.

4-The normal component of the electric flux is discontinuous if $\rho_S = 0$.

Interface between two lossless dielectric layers (no charge, no current at the interface) can be given as:



$$E_{1t} - E_{2t} = 0$$

$$H_{1t} - H_{2t} = J_S = 0$$

$$D_{1n} - D_{2n} = \rho_S = 0$$

$$B_{1n} - B_{2n} = 0$$

Example

Two extensive homogeneous isotropic dielectrics meet on plane $z = 0$. For $z \geq 0$, $\varepsilon_{r1} = 4$ and for $z \leq 0$, $\varepsilon_{r2} = 3$. A uniform electric field $\mathbf{E}_1 = 5\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z$ kV/m exists for $z \geq 0$. Find

(a) $\mathbf{E}_2$ for $z \leq 0$

(b) The angles E_1 and E_2 make with the interface

(c) The energy densities in J/m^3 in both dielectrics

Solution

Let the problem be as illustrated in Figure 5.15.

(a) Since $\mathbf{a}_z$ is normal to the boundary plane, we obtain the normal components as

$$E_{1n} = \mathbf{E}_1 \cdot \mathbf{a}_n = \mathbf{E}_1 \cdot \mathbf{a}_z = 3$$
$$\mathbf{E}_{1n} = 3\mathbf{a}_z$$
$$\mathbf{E}_{2n} = (\mathbf{E}_2 \cdot \mathbf{a}_z)\mathbf{a}_z$$

Also

$$\mathbf{E} = \mathbf{E}_n + \mathbf{E}_t$$

Hence,

$$\mathbf{E}_{1t} = \mathbf{E}_1 - \mathbf{E}_{1n} = 5\mathbf{a}_x - 2\mathbf{a}_y$$

Thus

$$\mathbf{E}_{2t} = \mathbf{E}_{1t} = 5\mathbf{a}_x - 2\mathbf{a}_y$$

Similarly,

$$\mathbf{D}_{2n} = \mathbf{D}_{1n} \rightarrow \varepsilon_{r2}\mathbf{E}_{2n} = \varepsilon_{r1}\mathbf{E}_{1n}$$

or

$$\mathbf{E}_{2n} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}\mathbf{E}_{1n} = \frac{4}{3}(3\mathbf{a}_z) = 4\mathbf{a}_z$$

Thus

$$\begin{aligned}\mathbf{E}_2 &= \mathbf{E}_{2t} + \mathbf{E}_{2n} \\ &= 5\mathbf{a}_x - 2\mathbf{a}_y + 4\mathbf{a}_z \text{ kV/m}\end{aligned}$$

(b) Let α_1 and α_2 be the angles $\mathbf{E}_1$ and $\mathbf{E}_2$ make with the interface while θ_1 and θ_2 are the angles they make with the normal to the interface as shown in Figures 5.15; that is,

$$\alpha_1 = 90 - \theta_1$$

$$\alpha_2 = 90 - \theta_2$$

Since $E_{1n} = 3$ and $E_{1t} = \sqrt{25 + 4} = \sqrt{29}$

$$\tan \theta_1 = \frac{E_{1t}}{E_{1n}} = \frac{\sqrt{29}}{3} = 1.795 \rightarrow \theta_1 = 60.9^\circ$$

Hence,

$$\alpha_1 = 29.1^\circ$$

Alternatively,

$$\mathbf{E}_1 \cdot \mathbf{a}_n = |\mathbf{E}_1| \cdot 1 \cdot \cos \theta_1$$

or

$$\cos \theta_1 = \frac{3}{\sqrt{38}} = 0.4867 \rightarrow \theta_1 = 60.9^\circ$$

Similarly,

$$E_{2n} = 4 \qquad E_{2t} = E_{1t} = \sqrt{29}$$

$$\tan \theta_2 = \frac{E_{2t}}{E_{2n}} = \frac{\sqrt{29}}{4} = 1.346 \rightarrow \theta_2 = 53.4^\circ$$

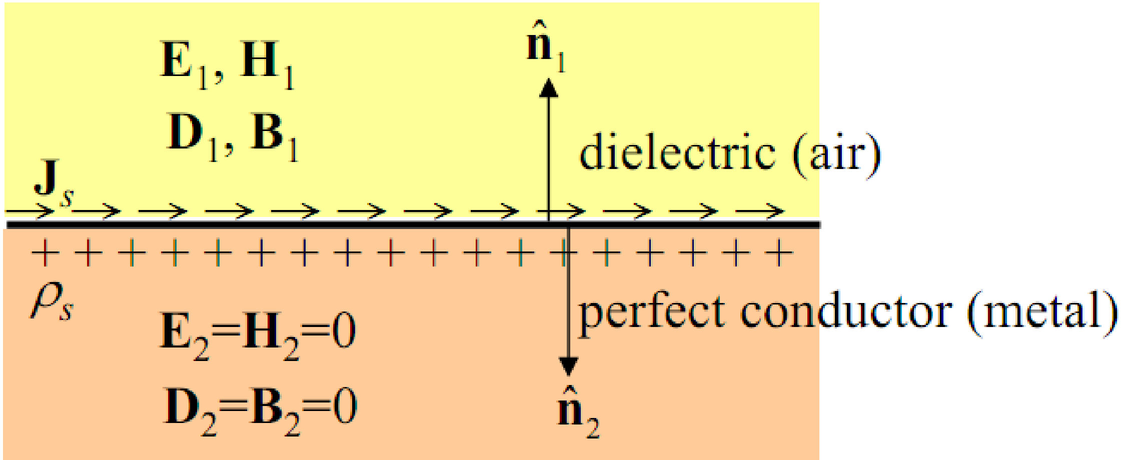
Hence,

$$\alpha_2 = 36.6^\circ$$

Note that $\dfrac{\tan \theta_1}{\tan \theta_2} = \dfrac{\varepsilon_{r1}}{\varepsilon_{r2}}$ is satisfied.

2- Conductive- Dielectric Boundary Conditions

The interface between a perfect conductor and a dielectic shown in figure



$$\hat{\mathbf{n}}_1 \times \mathbf{E}_1 = 0$$

$$\hat{\mathbf{n}}_1 \times \mathbf{H}_1 = \mathbf{J}_s$$

$$\hat{\mathbf{n}}_1 \cdot \mathbf{B}_1 = 0$$

$$\hat{\mathbf{n}}_1 \cdot \mathbf{D}_1 = \rho_s$$

The conductor is assumed to be perfect (i.e., $\sigma \rightarrow \infty$ or $\rho_s \rightarrow 0$). Although such a conductor is not practically realizable, we may regard conductors such as copper and silver as though they were perfect conductors.

To determine the boundary conditions for a conductor-dielectric interface, we incorporate the fact that E = 0 inside the conductor.

$$D_n = \rho_s$$

Thus under static conditions, the following conclusions can be made about a perfect conductor:

1. No electric field may exist within a conductor; that is

$$\boxed{\rho_s = 0 \qquad E = 0}$$

2. Since E =0, there can be no potential difference between any two points in the conductor; that is, a conductor is an equipotential body.

$$\boxed{D_t = \varepsilon_o\varepsilon_r E_t = 0, \qquad D_n = \varepsilon_o\varepsilon_r E_n = \rho_S}$$

3. The electric field E can be external to the conductor and normal to its surface; that is An important application of the fact that E = 0 inside a conductor is in electrostatic screening or shielding.

Example

Region $y \leq 0$ consists of a perfect conductor while region $y \geq 0$ is a dielectric medium ($\varepsilon_{1r} = 2$) as in Figure 5.16. If there is a surface charge of 2 nC/m^2 on the conductor, determine **E** and **D** at

(a) $A(3, -2, 2)$

(b) $B(-4, 1, 5)$

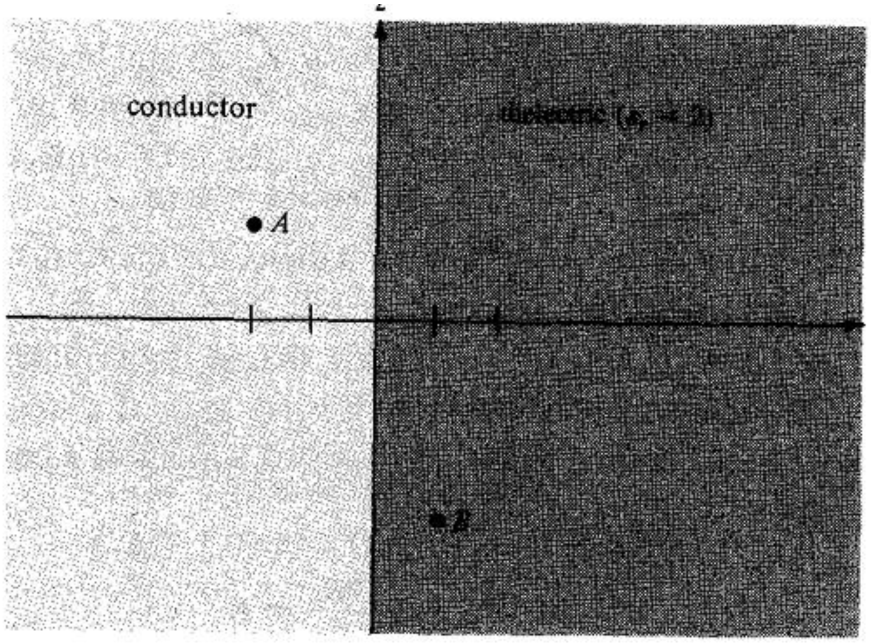
Solution

(a) Point $A(3, -2, 2)$ is in the conductor since $y = -2 < 0$ at A. Hence,

$$\mathbf{E} = 0 = \mathbf{D}$$

(b) Point $B(-4, 1, 5)$ is in the dielectric medium since $y = 1 > 0$ at B.

$$D_n = \rho_S = 2 \text{ nC/m}^2$$



Hence,

$$\mathbf{D} = 2\mathbf{a}_y \text{ nC/m}^2$$

and

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_o \varepsilon_r} = 2 \times 10^{-9} \times \frac{36\pi}{2} \times 10^9 \, \mathbf{a}_y = 36\pi\mathbf{a}_y$$
$$= 113.1\mathbf{a}_y \text{ V/m}$$