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Fluid Mechanics - Turbulence

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Contents:

- Transition, Reynolds averaging
- **Turbulent boundary and shear layers**
- Mixing-length models of turbulence
- One- and Two-equation models

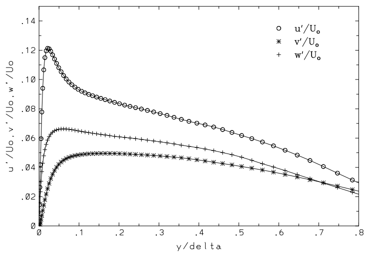
Reading:
J. Mathieu, J. Scott, *An Introduction to Turbulent Flow*
S.B. Pope, *Turbulent Flows*

Turbulent Boundary Layers

- We have seen that for steady, turbulent, incompressible flow the Reynolds averaged momentum equations are

$$\frac{\partial}{\partial x_j}(U_jU_i) = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial U_i}{\partial x_j} - \overline{u_iu_j}\right) \qquad (1)$$

- In a turbulent boundary layer the rms turbulent velocities are typically in the order of 10% or less of local mean velocities.
- This indicates that the Reynolds stresses $\overline{u_iu_j}$ are only a few percent of the mean velocity squared. The turbulent kinetic energy, defined as $k = (1/2)(\overline{u^2} + \overline{v^2} + \overline{w^2})$, is thus much less than that of the mean flow.



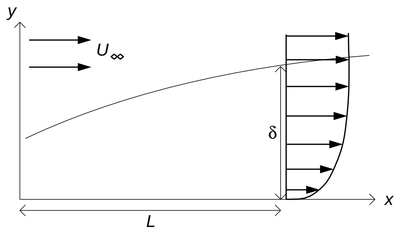
- Despite the above comment, one cannot neglect the turbulence. The appearance of the Reynolds stresses is the only difference between the equations governing a laminar flow and those governing the averaged velocities in a turbulent flow.
- In order to gain some understanding of the contribution of the Reynolds stresses to the momentum equations we consider the balance of the various terms in a simple boundary layer, and the rates of change of stresses across it.

- For this analysis we revert to ordinary Cartesian coordinates because rates of change in a boundary layer are very different in different directions.
- For a 2-D boundary layer with zero pressure gradient, the continuity and streamwise momentum equations reduce to:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \qquad (2)$$

$$\frac{\partial UU}{\partial x} + \frac{\partial VU}{\partial y} = \frac{\partial}{\partial x}\left(\nu\frac{\partial U}{\partial x} - \overline{u^2}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y} - \overline{uv}\right) \qquad (3)$$

- Over a distance L the boundary layer grows to a thickness δ, and we assume that $\delta \ll L$.
- If U is of order U_∞, then $\partial U/\partial x$ can be argued to be of order U_∞/L.
- Hence, in order for the two terms in the continuity equation to balance, V must be of order $U_\infty\delta/L$.



- Using the above estimates, the relative magnitudes of the terms in the U momentum equation are thus:

$$\underbrace{\frac{\partial UU}{\partial x} + \frac{\partial VU}{\partial y}}_{\frac{U_\infty^2}{L}} = \underbrace{\frac{\partial}{\partial x}\left(\nu\frac{\partial U}{\partial x}\right)}_{\frac{\nu U_\infty}{L^2}} - \underbrace{\frac{\partial}{\partial x}\left(\overline{u^2}\right)}_{\frac{(u')^2}{L}} + \underbrace{\frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y}\right)}_{\frac{\nu U_\infty}{\delta^2}} - \underbrace{\frac{\partial}{\partial y}\left(\overline{uv}\right)}_{\frac{(u')^2}{\delta}}$$

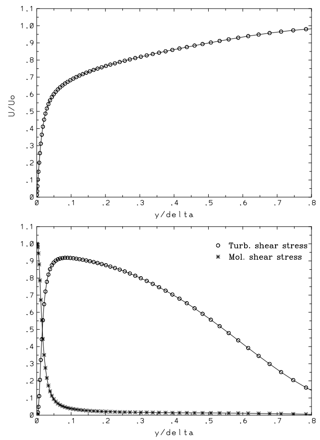
where u' is the typical scale of the turbulent velocity fluctuations.

- If we rescale all terms by multiplying each by L/U_∞^2, the relative magnitudes are then

$$1 \qquad \frac{\nu}{U_\infty L} \qquad \left(\frac{u'}{U_\infty}\right)^2 \qquad \left(\frac{L}{\delta}\right)^2\frac{\nu}{U_\infty L} \qquad \left(\frac{u'}{U_\infty}\right)^2\frac{L}{\delta}$$

- $U_\infty L/\nu$ is the Reynolds number based on length L and free-stream velocity U_∞.
- Since boundary layers become turbulent at Reynolds numbers of 10^6 or greater, the contribution of the viscous stresses is, on average, very small across the layer.
- We have also seen that $(u'/U_\infty)^2$ is small, with the data shown earlier suggesting it is of order 10^{-2}.
- However, in an equation, the left hand side must balance the right hand side, and thus we cannot have one term that is significantly greater than all the others.

- We can conclude, therefore, that the most influential term on the right hand side is $\partial\overline{uv}/\partial y$.
- In order for this term to be of the correct magnitude, we also conclude that $\delta \approx 10^{-2}L$.
- Notice that the streamwise gradient $\partial\overline{u^2}/\partial x$ is negligible in such a boundary layer, and it is only the *turbulent shear stress* $\overline{uv}$ which affects the mean velocity profile.
- However, the above analysis suggesting that viscous effects can be neglected cannot be correct across *all* of the boundary layer; in particular immediately adjacent to the wall, since the turbulent velocities must vanish at a rigid surface.
- As will be seen later, $\overline{uv} \propto y^3$ at the wall, so that in the immediate vicinity of the wall momentum must be diffused by viscosity rather than turbulent mixing.
- We thus get the picture of a near-wall viscosity-affected layer (often referred to as the viscous sublayer) where both viscous and turbulent effects may be important, and an outer, "fully turbulent" region, where direct viscous effects are negligible.



The y Momentum Equation in a Boundary Layer

- The cross-stream momentum equation is

$$\underbrace{\frac{\partial UV}{\partial x} + \frac{\partial}{\partial y}(VV)}_{\frac{U_\infty^2\delta}{L^2}} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial y}}_{??} + \underbrace{\frac{\partial}{\partial x}\left(\nu\frac{\partial V}{\partial x}\right)}_{\frac{\nu U_\infty \delta}{L^3}} - \underbrace{\frac{\partial}{\partial x}(\overline{uv})}_{\frac{(u')^2}{L}} + \underbrace{\frac{\partial}{\partial y}\left(\nu\frac{\partial V}{\partial y}\right)}_{\frac{\nu U_\infty}{\delta L}} - \underbrace{\frac{\partial}{\partial y}\left(\overline{v^2}\right)}_{\frac{(u')^2}{\delta}}$$

- The relative magnitudes of the terms are thus

$$1 \qquad ?? \qquad \frac{\nu}{U_\infty L} \qquad \left(\frac{u'}{U_\infty}\right)^2\frac{L}{\delta} \qquad \frac{\nu}{U_\infty L}\left(\frac{L}{\delta}\right)^2 \qquad \left(\frac{u'}{U_\infty}\frac{L}{\delta}\right)^2$$

- Since we have already concluded that $(u'/U_\infty)^2 \approx 10^{-2}$ and $\delta/L \approx 10^{-2}$, the term $\partial(\overline{uv})/\partial x$ is O(1) and the final term in the equation has magnitude 10^2.
- Thus, of all the terms whose relative magnitude is already known, the term $\partial\overline{v^2}/\partial y$ is the largest one.
- In an *equation* we cannot have one term much greater than all the others. Consequently, this last term must be balanced by the pressure gradient in the y direction:

$$\frac{1}{\rho}\frac{\partial P}{\partial y} \approx -\frac{\partial\overline{v^2}}{\partial y}$$

- Integrating this equation across the boundary layer from some point within it to the free stream (where turbulence is assumed to be negligible), we get:

$$P - P_\infty = -\rho\overline{v^2} \qquad \text{or} \qquad P = P_\infty - \rho\overline{v^2} \tag{4}$$

- Thus, in a simple turbulent boundary layer we have $P + \rho\overline{v^2}$ being constant across the boundary layer.
- Note the contrast between this and the corresponding laminar situation where P is constant across the boundary layer.

Total Shear Stress Across the Boundary Layer

- From the graph showing turbulent and molecular shear stress across the boundary layer one might guess that, across the near-wall layer, the sum of the two is almost constant.
- This result can be shown to follow from making certain assumptions about the boundary layer.
- Since streamwise gradients are small, the U-momentum equation in a zero pressure gradient boundary layer can be written

$$\frac{\partial(UU)}{\partial x} + \frac{\partial(VU)}{\partial y} = \frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y} - \overline{uv}\right) \tag{5}$$

- If convective effects are also assumed to be small, then the above equation reduces to

$$0 = \frac{\partial}{\partial y}\left(\nu\frac{\partial U}{\partial y} - \overline{uv}\right) \tag{6}$$

- Integrating this gives

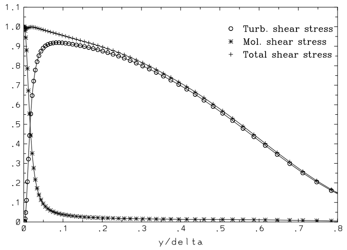
$$\nu\frac{\partial U}{\partial y} - \overline{uv} = Constant \tag{7}$$

- However, at the wall $\overline{uv} = 0$ and $\nu\partial U/\partial y$ is simply the wall shear stress τ_w/ρ.

- Hence we get the result that

$$\nu \frac{\partial U}{\partial y} - \overline{uv} = \tau_w / \rho \tag{8}$$

- Since $\nu \partial U / \partial y$ is the viscous shear stress and $-\overline{uv}$ the turbulent shear stress, we thus deduce that the total shear stress is constant across the layer and equal to the wall shear stress.
- As can be seen from the graph, the assumptions do not hold right across the boundary layer, but are a reasonable approximation across the near-wall part of it.



Mean Kinetic Energy Balance

- The mean kinetic energy K is defined as $K = 0.5(U^2 + V^2 + W^2)$. In tensor notation this is usually written as $K = 0.5U_i^2$.
- The transport equation for K can be obtained by multiplying the Reynolds equation for U_i by the velocity U_i (note that this implies summation over the index i):

$$U_i \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right] = U_i \left[-\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} - \frac{\partial \overline{u_i u_j}}{\partial x_j} \right]$$

- The terms on the left hand side give

$$U_i \frac{\partial U_i}{\partial t} = \frac{\partial}{\partial t} (U_i^2/2) \equiv \frac{\partial K}{\partial t} \quad \text{and} \quad U_i U_j \frac{\partial U_i}{\partial x_j} = U_j \frac{\partial}{\partial x_j} (U_i^2/2) \equiv U_j \frac{\partial K}{\partial x_j}$$

- The viscous terms can be written

$$\nu U_i \frac{\partial^2 U_i}{\partial x_j^2} = \frac{\partial}{\partial x_j} \left(U_i \nu \frac{\partial U_i}{\partial x_j} \right) - \nu \left(\frac{\partial U_i}{\partial x_j} \right)^2 = \frac{\partial}{\partial x_j} \left[\nu \frac{\partial}{\partial x_j} (U_i^2/2) \right] - \nu \left(\frac{\partial U_i}{\partial x_j} \right)^2$$

- The terms involving the Reynolds stresses are

$$-U_i \frac{\partial \overline{u_i u_j}}{\partial x_j} = \frac{\partial}{\partial x_j} (-U_i \overline{u_i u_j}) + \overline{u_i u_j} \frac{\partial U_i}{\partial x_j}$$

- Thus, the K equation can be written as

$$\frac{DK}{Dt} = -\frac{1}{\rho} \frac{\partial}{\partial x_j} (PU_j) + \frac{\partial}{\partial x_j} \left(\nu \frac{\partial K}{\partial x_j} - U_i \overline{u_i u_j} \right) + \overline{u_i u_j} \frac{\partial U_i}{\partial x_j} - \nu \left(\frac{\partial U_i}{\partial x_j} \right)^2 \tag{9}$$

- Note that $D\phi/Dt$ is used here, and elsewhere, to denote the total convective derivative:

$$\frac{D\phi}{Dt} \equiv \frac{\partial \phi}{\partial t} + U_j \frac{\partial \phi}{\partial x_j} \equiv \frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} + V \frac{\partial \phi}{\partial y} + W \frac{\partial \phi}{\partial z} \tag{10}$$

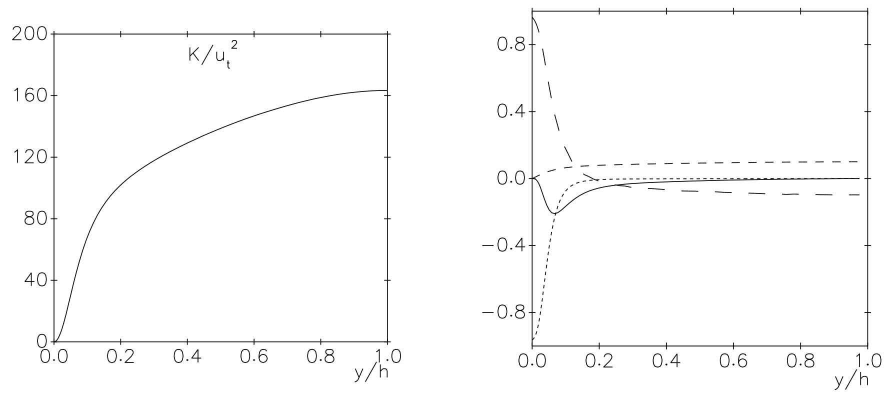
- Note also that the convection velocities can be written either inside or outside the derivatives, since in an incompressible flow

$$\frac{\partial U\phi}{\partial x} + \frac{\partial V\phi}{\partial y} + \frac{\partial W\phi}{\partial z} = U \frac{\partial \phi}{\partial x} + V \frac{\partial \phi}{\partial y} + W \frac{\partial \phi}{\partial z} + \phi \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} \right) \tag{11}$$

  and the term in brackets is zero from continuity.
- The first term on the right hand side of equation (9) is the pressure work.
- The second term is diffusive in character, representing mixing due to viscosity and turbulence.
- The final term must be negative, and represents dissipation of mean kinetic energy by viscous action.
- The term involving Reynolds stresses and mean velocity gradients links the mean and turbulent kinetic energy equations, as will be seen later.

Mean Kinetic Energy Budget in Plane Channel Flow

Mean kinetic energy | Mean kinetic energy budget



——: $\overline{uv}\, \partial U/\partial y$; – –: Pressure work;
— —: Diffusion; - - -: Viscous dissipation

(u_t is the friction velocity $(\tau_w/\rho)^{1/2}$, and h the channel half-height)

Turbulent Kinetic Energy Balance

- The turbulent kinetic energy k is defined by $k \equiv 0.5\overline{u_i^2} \equiv 0.5(\overline{u^2} + \overline{v^2} + \overline{w^2})$. To derive its transport equation, we can use

$$\frac{Dk}{Dt} = \frac{D}{Dt}\left(\overline{u_i^2}/2\right) = \overline{u_i \frac{Du_i}{Dt}} \qquad (12)$$

- To proceed further, we need a transport equation for the fluctuating velocity u_i. This can be obtained by subtracting the Reynolds-averaged momentum equation from the Navier Stokes equation:

$$\frac{Du_i}{Dt} = \frac{D\widetilde{U}_i}{Dt} - \frac{DU_i}{Dt}$$

- The derivation is left as an exercise, but the result is:

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - u_j \frac{\partial U_i}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2} - \frac{\partial}{\partial x_j}\left(u_i u_j - \overline{u_i u_j}\right) \qquad (13)$$

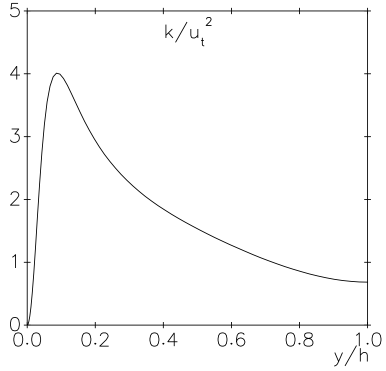
- Multiplying this equation by the fluctuating velocity u_i (note, again, the implied summation over the index i), and averaging, we can arrive at:

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} - \nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} - \frac{\partial}{\partial x_i}\left(\overline{u_j^2 u_i}/2 + \overline{u_i p}/\rho - \nu \frac{\partial k}{\partial x_i}\right) \qquad (14)$$

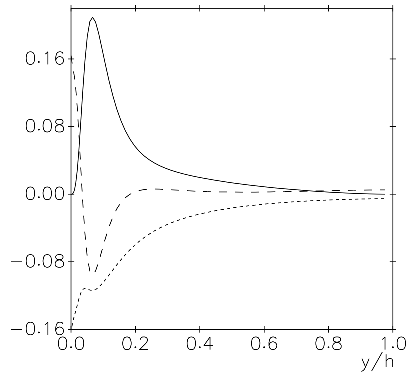
- The second term on the right hand side of equation (14) represents the dissipation of turbulent kinetic energy by viscosity at the smallest scales.
- The last term represents diffusion, as a result of viscous and turbulent mixing.
- The term $-\overline{u_i u_j}\,\partial U_i/\partial x_j$ must therefore represent the generation of turbulent kinetic energy. It appears in both kinetic energy equations, and can thus be interpreted as the rate at which kinetic energy is lost from the mean flow and transferred to the turbulent eddies.
- The term $-\overline{u_i u_j}\,\partial U_i/\partial x_j$ is often denoted by P_k and called the production, or generation, rate of k.
- In most circumstances P_k is positive, representing a transfer of kinetic energy from the mean flow to the turbulence. However, there are flow conditions under which P_k can be locally negative in certain regions.

Turbulent Kinetic Energy Budget in Plane Channel Flow

Turbulent kinetic energy



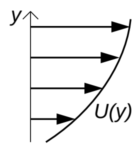
Turbulent kinetic energy budget



——: $-\overline{uv}\,\partial U/\partial y$;
– –: Diffusion;
- - -: Dissipation

Energy Flow Processes Near a Wall

- If we make the usual boundary layer or thin shear flow approximation that $U(y)$ is the only non-zero mean velocity component, then the mean kinetic energy equation becomes:



$$\frac{DK}{Dt} = \underbrace{\nu\frac{\partial^2 K}{\partial y^2}}_{\text{Term 1}} - \underbrace{\nu\left(\frac{\partial U}{\partial y}\right)^2}_{\text{Term 2}} - \underbrace{\frac{\partial}{\partial x}(PU/\rho)}_{\text{Term 3}} + \underbrace{\overline{uv}\frac{\partial U}{\partial y}}_{\text{Term 4}} - \underbrace{\frac{\partial}{\partial y}(U\overline{uv})}_{\text{Term 5}} \qquad (15)$$

- If we consider a simple turbulent near-wall flow, where the streamwise pressure gradient is small, the convective rate of change of K is generally found to be much smaller than the individual source and sink terms on the right.
- The K equation is then approximated by

$$0 = \underbrace{\nu\frac{\partial^2 K}{\partial y^2}}_{\text{Term 1}} - \underbrace{\nu\left(\frac{\partial U}{\partial y}\right)^2}_{\text{Term 2}} + \underbrace{\overline{uv}\frac{\partial U}{\partial y}}_{\text{Term 4}} - \underbrace{\frac{\partial}{\partial y}(U\overline{uv})}_{\text{Term 5}} \qquad (16)$$

- As seen earlier, in the near-wall layer of a zero pressure gradient turbulent boundary layer we get a constant total shear stress:

$$\nu \frac{\partial U}{\partial y} - \overline{uv} = Const = \tau_w/\rho \tag{17}$$

- This result showed that the sum of the turbulent shear stress (-$\overline{uv}$) and molecular shear stress ($\nu \partial U/\partial y$) is constant across the boundary layer (and equal in magnitude to the wall shear stress).
- As seen in the earlier graphs, very close to the wall the turbulent shear stress becomes small in magnitude and viscous effects must therefore grow. The viscous terms in the K equation will therefore dominate in this region, and Terms 1 and 2 in equation (16) must therefore be in balance very close to the wall.
- Beyond this viscous sublayer, however, the effects of viscosity on the mean flow are negligible. There must then be a balance between terms 4 and 5.
- In this outer region (the fully-turbulent region), we will see later that the mean velocity U varies as $\log(y)$, so $\partial U/\partial y \propto y^{-1}$.
- When viscous effects are negligible, equation (17) shows that the turbulent shear stress $\overline{uv} \approx -\tau_w/\rho$. The rate of loss of mean kinetic energy to turbulence (Term 4) is thus proportional to y^{-1} in this outer region, so decreases with distance from the wall.

- However, $\overline{uv}\, \partial U/\partial y$ is zero at the wall (as $\overline{uv}$ is zero there). Since, as noted above, it decreases in magnitude across the 'log-layer' as y increases, it must reach a maximum in magnitude at some point between the wall and the log-layer.
- Thus the rate of transforming mean kinetic energy into turbulent kinetic energy must be greatest at some point closer to the wall than the log-layer.
- We can examine where exactly in the boundary layer this energy transformation rate is greatest.
- Since the energy transfer rate is given by $-\overline{uv}\, \partial U/\partial y$, the maximum transfer rate occurs where

$$\frac{\partial}{\partial y}\left(\overline{uv}\, \frac{\partial U}{\partial y}\right) = 0 \tag{18}$$

- This equation can be expanded to give

$$\frac{\partial \overline{uv}}{\partial y} \frac{\partial U}{\partial y} + \overline{uv}\, \frac{\partial^2 U}{\partial y^2} = 0 \tag{19}$$

- Eliminating $\overline{uv}$ from the first term, with the help of equation (17), we obtain

$$\frac{\partial U}{\partial y} \frac{\partial}{\partial y}\left(\nu \frac{\partial U}{\partial y} - \tau_w/\rho\right) + \overline{uv}\, \frac{\partial^2 U}{\partial y^2} = 0$$

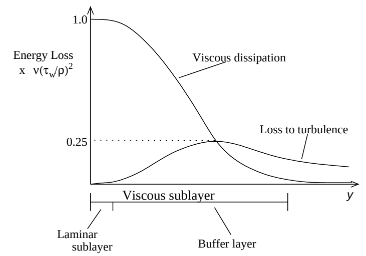
- Since ν and τ_w/ρ are constants:

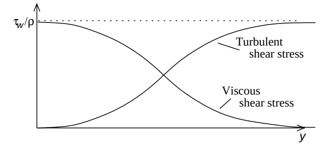
$$\nu \frac{\partial U}{\partial y} \frac{\partial^2 U}{\partial y^2} = -\overline{uv}\, \frac{\partial^2 U}{\partial y^2}$$

- Finally, on cancelling $\partial^2 U/\partial y^2$:

$$\nu \frac{\partial U}{\partial y} = -\overline{uv}$$

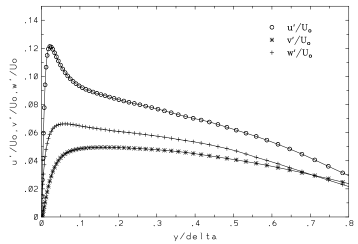
- Hence the rate of loss of mean kinetic energy to turbulence is greatest where the viscous stress equals the turbulent shear stress.
- The generation rate of k thus takes its maximum not in the 'fully turbulent' region, but in the viscosity-affected region of the boundary layer.
- It is also easy to show that at this point the viscous dissipation of mean kinetic energy equals the loss to turbulence.





Near-Wall Reynolds Stresses

- As seen earlier, there are significant differences between the levels of the stress components away from the wall. In a boundary layer (or shear flow) with mean velocity $U(y)$, one generally finds $\overline{u^2} > \overline{w^2} > \overline{v^2}$.
- The question we address here is how do the stresses behave very close to the wall, as they approach zero.



- To examine the near-wall behaviour of the stresses, we can express the near-wall velocities as Taylor series expansions in powers of y (the wall-normal distance):

$$u = a_1 y + b_1 y^2 + c_1 y^3 + \ldots$$
$$v = a_2 y + b_2 y^2 + c_2 y^3 + \ldots$$
$$w = a_3 y + b_3 y^2 + c_3 y^3 + \ldots$$

where the a's, b's, c's etc. are functions of x, z and t.

- The above expansions do ensure that the velocities vanish at the wall, but they must also satisfy continuity ($\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$).

- Substituting the expansions for the velocities into the continuity equation:

$$\begin{array}{l} y\,\partial a_1/\partial x \quad + \quad y^2\,\partial b_1/\partial x \quad + \quad \ldots \\ + \quad a_2 \quad + \quad 2\,y\,b_2 \quad + \quad 3\,y^2\,c_2 \quad + \quad \ldots \\ + \quad y\,\partial a_3/\partial z \quad + \quad y^2\,\partial b_3/\partial z \quad + \quad \ldots = 0 \end{array}$$

- Considering the $O(1)$ terms leads to $a_2 = 0$.

- Hence close to the wall the velocities behave as $u \propto y$, $w \propto y$, but $v \propto y^2$.

- The Reynolds stresses thus behave as $\overline{u^2} \propto y^2$, $\overline{v^2} \propto y^4$, $\overline{w^2} \propto y^2$ and $\overline{uv} \propto y^3$ for small y.