TRANSMISSION LINES AND WAVEGUIDES

Uniformity along the Direction of Propagation

Definition: Transmission Line

TL is the term to describe transmission systems with two or more metallic conductors electrically insulated from each other.

Definition: Waveguide

The propagation in a waveguide is generally ensured by successive reflections on the guide boundaries. These are conducting walls in the case of metallic WG's. Dielectric WG's and optical fibers utilize the total internal reflection.

Clasification of the Modes of Propagation

The presence and apsence of longitudial field components affects the propagation behaviour of the modes. Four mode categories can exist:

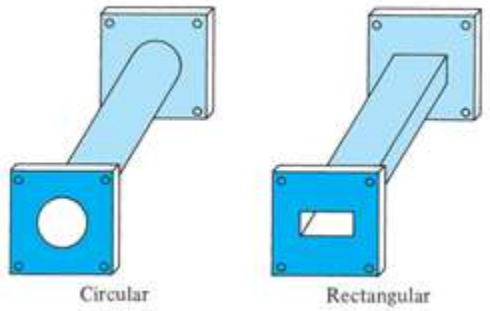
E_z	H_z	Name	Mode
0	0	Transverse Electromagnetic	TEM
0	$\neq 0$	Transverse Electric	TE
$\neq 0$	0	Transverse Magnetic	TM
$\neq 0$	$\neq 0$	Hybrid	

Classification of Lines and Waveguides

A large number of different structures can be used to transmit electromagnetic signals. They can be:

1) Either open (radiation can take place) or closed (fields enclosed within a conducting envelope);
2) Either homogeneous (one single propagation medium without the transverse dependence of the material properties) or inhomogeneous (several different propagation media);
3) Either conductorless or processing one or more conductors.

PROPAGATION WITHIN A CLOSED HOMOGENEOUS METALLIC WAVEGUIDE



We make the following assumptions:

1) The structure is uniform along the direction of propagation.
2) The cross-section is arbitrary. It can be simply connected (hollow WG) or multiply connected (several conductors).
3) The material which completely fills the WG is isotropic, linear and homogeneous. Its material parameters ε, μ and σ do not depend on the position within the WG, nor upon the amplitude of the signal.
4) The WG does not contain any electrical space charge $(\rho = 0)$.

Separation of Maxwell's Equations into Longitudinal and Transverse Components

The del operator ∇ can be expressed as:

$$\nabla = \nabla_t + \hat{a}_z \frac{\partial}{\partial z}$$

Where ∇_t is the transverse del operator and is given in the Cartesian coordinates by

$$\nabla_t = \hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y}$$

Assuming now, time-harmonic fields with an $e^{j\omega t}$ time dependence and wave propagation along the + z-axis, the field vectors can be written as:

$$\bar{E}(x,y,z)=\left[\bar{e}(x,y)+\hat{a}_z e_z(x,y)\right]e^{-j\beta z}$$
$$\bar{H}(x,y,z)=\left[\bar{h}(x,y)+\hat{a}_z h_z(x,y)\right]e^{-j\beta z}$$

The equation

$$\nabla X\,\bar{E}=-j\omega\mu\bar{H}$$

Will take the form:

$$\left(\nabla_t+\hat{a}_z\frac{\partial}{\partial z}\right)X\left[\left[\bar{e}(x,y)+\hat{a}_z e_z(x,y)\right]e^{-j\beta z}\right]=$$
$$-j\omega\mu\,\bar{h}(x,y)e^{-j\beta z}-j\omega\mu\,h_z(x,y)e^{-j\beta z}\hat{a}_z$$

Or

$$\left(\nabla_t X\,\bar{e}\right)e^{-j\beta z}+\nabla_t X(\hat{a}_z e_z\,e^{-j\beta z})+\hat{a}_z\frac{\partial}{\partial z}X(\bar{e}\,e^{-j\beta z})$$
$$+\hat{a}_z\frac{\partial}{\partial z}X\left(\hat{a}_z e_z\right)e^{-j\beta z}$$
$$=-j\omega\mu\bar{h}e^{-j\beta z}-j\omega\mu h_z e^{-j\beta z}\hat{a}_z$$

Or,

$$\left[\nabla_t X\,\bar{e}\right]e^{-j\beta z}+\hat{a}_z X\left[-j\beta\bar{e}e^{-j\beta z}-\nabla_t e_z e^{-j\beta z}\right]$$
$$=-j\omega\mu\bar{h}e^{-j\beta z}-j\omega\mu h_z e^{-j\beta z}\hat{a}_z$$

From the second curl equation $\nabla X\bar{H}=\sigma\bar{E}+j\omega\varepsilon\bar{E}$ we equivalently get:

$$\left[\nabla_t X\,\bar{h}\right]e^{-j\beta z}+\hat{a}_z X\left[-j\beta\bar{h}\,e^{-j\beta z}-\nabla_t h_z e^{-j\beta z}\right]$$
$$=\left(\sigma+j\omega\varepsilon\right)\bar{e}\,e^{-j\beta z}+\left(\sigma+j\omega\varepsilon\right)e_z e^{-j\beta z}\hat{a}_z$$

Equating now the transverse and longitudinal parts gives:

$$\nabla_t X\,\bar{e}\left(x,y\right)=-j\omega\mu h_z\left(x,y\right)\hat{a}_z$$
$$\nabla_t X\,\bar{h}\left(x,y\right)=\left(\alpha+j\omega\varepsilon\right)e_z\left(x,y\right)\hat{a}_z$$ Longitudinal Parts

$$\nabla_t e_z+j\beta\bar{e}\left(x,y\right)=-j\omega\mu\,\hat{a}_z X\,\bar{h}\left(x,y\right)$$
$$\nabla_t h_z+j\beta\bar{h}\left(x,y\right)=(\sigma+j\omega\varepsilon)\,\hat{a}_z X\,\bar{e}\left(x,y\right)$$ Transverse Parts

Expanding the operators, we get the following scalar equations:

$$\frac{\partial e_z}{\partial y}+j\beta e_y=-j\omega\mu h_x \qquad \frac{\partial h_z}{\partial y}+j\beta h_y=j\omega\varepsilon e_x$$
$$-j\beta e_x-\frac{\partial e_z}{\partial x}=-j\omega\mu h_y \qquad -j\beta h_x-\frac{\partial h_z}{\partial x}=j\omega\varepsilon e_y$$
$$\frac{\partial e_y}{\partial x}-\frac{\partial e_x}{\partial y}=-j\omega\mu h_z \qquad \frac{\partial h_y}{\partial x}-\frac{\partial h_x}{\partial y}=j\omega\varepsilon e_z$$

Above we assumed that $\sigma = 0$ (lossless dialectic in a WG).

GENERAL SOLUTIONS FOR THE TEM, TE AND TM WAVES

The above six equations can be solved for the four transverse field components in terms of e_z and h_z as:

$$h_x = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial e_z}{\partial y} - \beta\frac{\partial h_z}{\partial x}\right)$$

$$h_y = -\frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial e_z}{\partial x} + \beta\frac{\partial h_z}{\partial y}\right)$$

$$e_x = -\frac{j}{k_c^2}\left(\beta\frac{\partial e_z}{\partial x} + \omega\mu\frac{\partial h_z}{\partial y}\right)$$

$$e_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial e_z}{\partial y} + \omega\mu\frac{\partial h_z}{\partial x}\right)$$

Where $k_c^2 = k^2 - \beta^2$ and $k = \omega\sqrt{\varepsilon\mu}$.

TEM WAVES

For TEM waves, the fields are, by definition, purely transverse (the field vectors $\bar{E}$ and $\bar{H}$ lie in the plane

which is perpendicular to the direction of propagation). As a result, E_z and H_z are both zero. Then $\nabla.\bar{E}=0$ gives:

$$\left(\nabla_t+\hat{a}_z\frac{\partial}{\partial z}\right).\left(\bar{e}\left(x,y\right)+\hat{a}_z e_z\left(x,y\right)\right)e^{-j\beta z}=0$$

Or,

$$\nabla_t.\bar{e}(x,y)-j\beta e_z=0$$

$$\nabla_t.\bar{e}(x,y)=j\beta e_z$$

Similarly, from $\nabla.\bar{H}=0$, we obtain

$$\nabla_t.\bar{h}(x,y)=j\beta h_z$$

Since $e_z=0$, $h_z=0$ for a TEM wave, we have

$$\nabla_t.\bar{e}\left(x,y\right)=0 \qquad \nabla_t.\bar{h}\left(x,y\right)=0$$

Also, from longitudinal parts we write

$$\nabla_t X\,\bar{e}\left(x,y\right)=0 \qquad \nabla_t X\,\bar{h}\left(x,y\right)=0$$

The equations

$$\nabla_t X\,\bar{e}\left(x,y\right)=0 \qquad \nabla_t.\,\bar{e}\left(x,y\right)=0$$

Define a two-dimensional electrostatic problem while

$$\nabla_t X\,\bar{h}\left(x,y\right)=0 \qquad \nabla_t.\,\bar{h}\left(x,y\right)=0$$

describe a two-dimensional magnetostatic problem.

Consider now,

$$\frac{\partial e_z}{\partial y} + j\beta e_y = -j\omega\mu h_x$$

$$-j\beta h_x - \frac{\partial h_z}{\partial x} = j\omega\varepsilon e_y$$

Which reduces to

$$j\beta e_y = -j\omega\mu h_x$$

$$-j\beta h_x = j\omega\varepsilon e_y$$

$$j\beta e_y = -j\omega\mu\left(\frac{j\omega\varepsilon}{-j\beta}\right)e_y \;\rightarrow\; j\beta e_y = j\frac{\omega^2\varepsilon\mu}{\beta}e_y$$

Or,

From which we conclude that $\beta = \omega\sqrt{\varepsilon\mu} = k$

In this case $k_c = 0$.

Since $\nabla^2 E_x + k^2 E_x = 0$ inside the WG and $E_x = e_x e^{-j\beta z}$

We will have $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)e_x e^{-j\beta z} + k^2 e_x e^{-j\beta z} = 0$

Or

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \beta^2 e_x + k^2 e_x = 0$$

Since $\beta = k$ we get,

$$\nabla_t^2 e_x = 0$$

We can write $\nabla_t^2 \overline{e} = 0$, $\nabla_t^2 \overline{h} = 0$

Both problems admit as solutions those of the two-dimensional Laplace's equation.

For a simply connected WG, Laplace's equation admits the trivial solution $\overline{e}(x, y) = 0$. The boundary condition imposes a constant potential on the metal tube. Solutions of Laplace's equation require that the potential be constant everywhere inside the tube. Thus, <u>an empty hollow WG cannot propagate a TEM wave.</u>

On a two-conductor structure, a TEM wave can propagate, since different potentials may exist on the two conductors. The voltage between two conductors can be found as:

$$V_{12} = \int_1^2 \overline{e}.d\overline{l}$$

The current on a conductor can be found from Ampere's Law as:

$$I = \oint_c \overline{H}.d\overline{l}$$

Where *C,* is the cross-sectional contour.

The wave impedance of a TEM mode is defined as:

$$Z_{TEM} = \frac{e_x}{h_y} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

$$Z_{TEM} = -\frac{e_y}{h_x} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

The propagation of a TEM mode on a homogeneous multiconductor line only depends on the propagation medium; it is independent of the geometry and the line dimensions.
The TEM mode can propagate at any frequency on a multiconductor.

For a TEM mode, we have,

$$\overline{h}(x,y) = \frac{1}{Z_{TEM}} \hat{a}_z X \, \overline{e}(x,y)$$

<u>TE WAVES</u>

$E_z = 0$ (by definition), $H_z \neq 0$. Then,

$$h_x = -\frac{j\beta}{k_c^2}\frac{\partial h_z}{\partial x}$$

$$h_y = -\frac{j\beta}{k_c^2}\frac{\partial h_z}{\partial y}$$

$$e_x = -\frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial y}$$

$$e_y = \frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial x}$$

Where, h_z satisfies:

$$\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial x^2}+k_c^2\right)h_z=0, \qquad k_c^2=k^2-\beta^2$$

The solutions of the above equation are only found for particular values of k_c, when we apply the boundary conditions. These are the eigenvalues of the TE mode problem. The transverse wavenumber k_c is specified by the guide cross-section (shape and size) and by the transverse distribution of the fields for the mode considered; it is independent of the medium filling the guide. In a homogeneous guide, the transverse wave number is always real and positive.

The transverse fields can compactly be written as:

$$\bar{e}(x,y)=\frac{j\omega\mu}{k_c^2}\hat{a}_z X\nabla_t h(x,y), \quad \bar{h}(x,y)=-\frac{j\beta}{k_c^2}\nabla_t h_z$$

The TE wave impedance is:

$$Z_{TE}=\frac{E_x}{H_y}=-\frac{E_y}{H_x}=\frac{\omega\mu}{\beta}=\frac{k\eta}{\beta}$$

Since β is frequency dependent, Z_{TE} depends on the frequency. TE waves can be supported inside closed conductors, as well as between two or more conductors.

TM WAVES

$E_z \neq 0$ (by definition), $H_z = 0$. Then,

$$h_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial e_z}{\partial y}$$

$$h_y = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial e_z}{\partial x}$$

$$e_x = -\frac{j\beta}{k_c^2}\frac{\partial e_z}{\partial x}$$

$$e_y = -\frac{j\beta}{k_c^2}\frac{\partial e_z}{\partial y}$$

Where e_z satisfies:

$$\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^2} + k_c^2\right)e_z = 0, \qquad k_c^2 = k^2 - \beta^2$$

The transverse fields can compactly be written as:

$$\bar{e}(x,y) = -\frac{j\beta}{k_c^2}\nabla_t e_z(x,y), \qquad \bar{h}(x,y) = -\frac{j\omega\varepsilon}{k_c^2}\hat{a}_z X \nabla_t e_z(x,y)$$

The TM Wave Impedance:

$$Z_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\beta}{\omega\varepsilon} = \frac{\beta\eta}{k} \quad (Frequency\ dependent)$$

TM waves can be supported inside hollow conductors, as well as between two conductors.

RECTANGULAR AND CIRCULAR WAVEGUIDES

Hollow WG's are commonly used as TL's at frequencies above *5GHz*. Compared to coaxial lines; WG's have the following advantages:

1) Higher power handling capability
2) Lower loss per unit length
3) A simpler, lower cost mechanical structure
4) The reflections caused by the flanges used in connecting WG sections is usually less than that associated with coaxial connectors.

The disadvantages are:

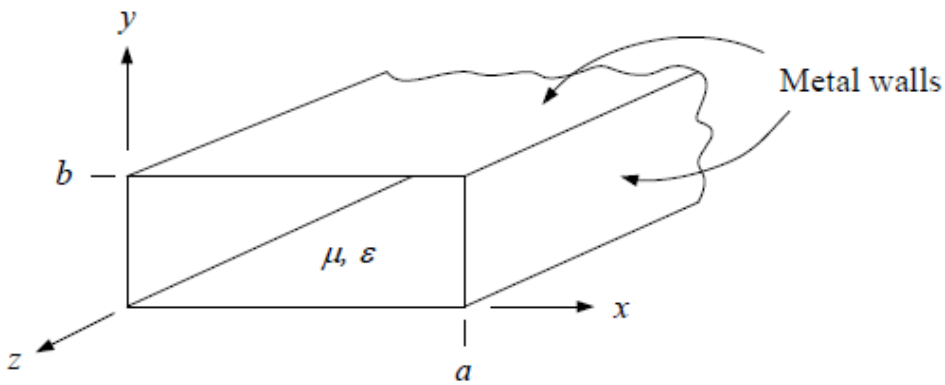
1) Lager cross-sectional dimensions
2) Lower usable bandwidth

A large variety of components such as couplers, detectors, isolators, attenuators and slotted lines are commercially available for various WG bands from *1GHz* to over *220GHz.*

The hollow WG's can support TM and TE modes but not TEM modes.

RECTANGULAR WAVEGUIDES

Conventionally, the longer side of the WG is located along the x-axis, so $a > b$.



<u>TE MODES</u>

$E_z = 0$ by definition and h_z satisfies:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) h_z(x, y) = 0 \text{ .....(*)}$$

The complete expression for $H_z(x, y)$ is:

$$H_z(x, y, z) = h_z(x, y) e^{-j\beta z}$$

And $k_c^2 = k^2 - \beta^2$ with $k^2 = \omega^2 \varepsilon \mu$.

A Separation of Variables Solution for $h_z(x,y)$

Assume $h_z(x,y) = X(x)Y(y)$

Substituting this into (*) and dividing each term by XY :

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + k_c^2 = 0$$

This equation can be satisfied for all x and y only if:

$$\frac{1}{X}\frac{d^2X}{dx^2} + k_x^2 = 0 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} + k_y^2 = 0$$

With:

$$k_x^2 + k_y^2 = k_c^2$$

The general solution for $h(x,y)$ can then is written as:

$$h_z(x,y) = \left(A\cos k_x x + B\sin k_x x\right)\left(C\cos k_y y + D\sin k_y y\right)$$

<u>The Boundary Conditions:</u>

We must have:

$e_x(x,y) = 0 \quad at \;\; y = 0$ and $y = b$

$e_y(x,y) = 0 \quad at \;\; x = 0$ and $x = a$

But we had,

$$e_x = -\frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial y} \qquad e_y = \frac{j\omega\mu}{k_c^2}\frac{\partial h_z}{\partial x}$$

Therefore it is necessary that:

We must have

$\frac{\partial h_z}{\partial y} = 0 \quad at \;\; y = 0$ and $y = b$

$\frac{\partial h_z}{\partial x} = 0 \quad at \;\; x = 0$ and $x = a$

Or

$-Ak_x \sin k_x x + Bk_x \cos k_x x = 0 \; at \;\; x = 0$ and $x = a$ which gives $B = 0$ for $k_x \neq 0$ and

$$-Ak_x \sin k_x a = 0$$

Which gives,

$$k_x a = m\pi \quad (m = 0,1,2,\ldots)$$

Similarly,

$-Ck_y \sin k_y y + Dk_y \cos k_y y = 0 \; at \;\; y = 0$ and $y = b$ which gives $D = 0$ for $k_y \neq 0$ and

$$-Ck_y \sin k_y b = 0$$

Which gives,

$$k_y b = n\pi \quad (n = 0,1,2,\ldots)$$

So, $k_x = \dfrac{m\pi}{a} \qquad k_y = \dfrac{n\pi}{b} \quad m = 0,1,2,\ldots \quad n = 0,1,2,\ldots$

Then,

$$h_z(x,y) = A_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

And

$$H_z(x,y,z) = A_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

Where A_{mn} : Constant depending on the excitation strength.

The other field components are found as:

$$E_x = \frac{j\omega\mu\pi}{k_c^2 b} nA_{mn}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$E_y = -\frac{j\omega\mu\pi}{k_c^2 a} mA_{mn}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$H_x = \frac{j\beta\pi}{k_c^2 a} mA_{mn}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$H_y = \frac{j\beta\pi}{k_c^2 b} nA_{mn}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

<u>The propagation constant</u>

$$\beta = \sqrt{k^2 - k_c^2}$$

Since $k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

$$\beta = \left[k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}}$$

$$\beta = \left[\omega^2\varepsilon\mu - \left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}\right]^{\frac{1}{2}}$$

Consider two cases of interest:

A) $k > k_c$

If the frequency *f* is high enough so that, for a given set of values of *a, b, m* and *n*,

$$k = \omega\sqrt{\varepsilon\mu} = 2\pi f\sqrt{\varepsilon\mu} > \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

then β is real. Real β corresponds to propagation. k_c is the cutoff wavenumber. Each mode has a cutoff frequency $f_{c_{mn}}$ given by:

$$f_{c_{mn}} = \frac{k_c}{2\pi\sqrt{\varepsilon\mu}} = \frac{1}{2\pi\sqrt{\varepsilon\mu}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

Let $v = \frac{1}{\sqrt{\varepsilon\mu}}$ =Phase velocity for an unbounded medium filled with material having ε and μ. Then,

$$f_{c_{mn}} = \frac{1}{2\pi\sqrt{\varepsilon\mu}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2} = \frac{v}{2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}$$

The mode with the <u>lowest cutoff frequency</u> is called the **dominant mode.**

Since *a>b*, the lowest f_c occurs for m=1 and *n=0*. So,

$$f_{c_{10}} = \frac{1}{2\pi\sqrt{\varepsilon\mu}}\frac{\pi}{a} = \frac{1}{2a\sqrt{\varepsilon\mu}} = \frac{v}{2a}$$

TE_{10} mode is the dominant mode.

Since E_x, E_y, H_x and H_y are all zero for *m*=*n*=0, there is no TE_{00} mode.

B) $k < k_c$

In this case β becomes purely imaginary,

$$\beta = -j\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2\right]^{1/2} = -jq$$, q is real. Then, the term $e^{-j\beta z}$ becomes $e^{-j(-jq)z} = e^{-qz}$ which corresponds to the attenuation of fields exponentially. Such modes are non-propagating or evanescent. Note that this attenuation is not associated with the dissipative losses.

<u>The Wave Impedance:</u>

$$Z_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{k\eta}{\beta}$$

$$k = \omega\sqrt{\varepsilon\mu}, \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

i) When $k > k_c$ $(f > f_c)$, β and Z_{TE} are both real.

ii) When $k < k_c$ $(f < f_c)$, β and Z_{TE} are both purely imaginary.

$$Z_{TE_{10}} = \frac{k\eta}{\beta_{10}} = \frac{k\eta}{\left[\omega^2\varepsilon\mu - \left(\frac{\pi}{a}\right)^2\right]^{1/2}} = \frac{\omega\mu}{\left[\omega^2\varepsilon\mu - \left(\frac{\pi}{a}\right)^2\right]^{1/2}}$$

<u>The guide wavelength:</u>

$$\lambda_g = \frac{2\pi}{\beta}$$

For a propagating mode $\beta = \sqrt{k^2 - k_c^2}$ so $\beta < k$ and

$$\lambda_g = \frac{2\pi}{\beta} > \lambda = \frac{2\pi}{k}$$

$$\lambda_g = 2\pi\left[k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{-1/2}$$

Let, $\lambda_c = \frac{2\pi}{k_c}$, , cutoff wavelength. Then,

$$\lambda_g = \frac{2\pi}{\sqrt{k^2 - k_c^2}} = \frac{1}{\left[\left(\frac{k}{2\pi}\right)^2 - \left(\frac{k_c}{2\pi}\right)^2\right]^{1/2}}$$

Or,

$$\lambda_g = \frac{1}{\left(\frac{1}{\lambda^2} - \frac{1}{\lambda_c^2}\right)^{1/2}} = \frac{1}{\frac{1}{\lambda}\left(1 - \frac{\lambda^2}{\lambda_c^2}\right)^{1/2}} = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{v/f}{\sqrt{1-(f_c/f)^2}}, \qquad v = \frac{1}{\sqrt{\varepsilon\mu}}$$

We have the following relationship:

$$\frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2} = \frac{1}{\lambda^2}$$

For the dominant mode:

$k_c = \frac{\pi}{a}$, $\lambda_c = \frac{2\pi}{\pi/a}$, so $\lambda_c = 2a$

$$\frac{1}{\lambda_g^2} = \frac{1}{\lambda^2} - \frac{1}{(2a)^2}$$

The phase velocity:

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{k^2 - k_c^2}} > \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}}$$

$$v_p > \frac{1}{\sqrt{\varepsilon\mu}}$$

$$f_c = \frac{k_c}{2\pi\sqrt{\varepsilon\mu}}$$

$$v_p = \omega\left[\omega^2\varepsilon\mu - \omega_c^2\varepsilon\mu\right]^{-1/2} = \frac{\omega}{\sqrt{\varepsilon\mu}}\frac{1}{\omega\sqrt{1-\left(\omega_c/\omega\right)^2}}$$

$$v_p = \frac{1/\sqrt{\varepsilon\mu}}{\sqrt{1-\left(f_c/f\right)^2}}$$