Dispersion for a Rectangular WG

Consider $\beta^2 = k^2 - k_c^2$.

i) If $f<f_c$, $\beta = \pm j\alpha$ and $\alpha^2 + k^2 = k_c^2$

$$\alpha = \sqrt{k_c^2 - k^2}$$

$$= \sqrt{\left(\frac{2\pi}{\lambda_c}\right)^2 - \left(\frac{2\pi}{\lambda}\right)^2}$$

$$= 2\pi\sqrt{\left(\frac{1}{\lambda_c}\right)^2 - \left(\frac{1}{\lambda}\right)^2}$$

$$= \frac{2\pi}{\lambda_c}\sqrt{1 - \left(\frac{k}{k_c}\right)^2}$$

$$\alpha = \frac{2\pi}{\lambda_c}\sqrt{1 - \left(\frac{f}{f_c}\right)^2}$$

ii) If $f>f_c$, β is real and $k^2 - \beta^2 = k_c^2$.

DEGENERATE MODES

In WG's, several modes with different configurations have the cutoff frequency. These are called degenerate modes. It can be seen that in a waveguide the possible TE_{mn} and TM_{mn} modes are always degenerate. The waveguide dimensions are always selected in a way that only the desired mode (generally TE_{10} or TM_{11}) propagate and higher modes are not supported.

TEH DOMINANT MODE (TE_{10} MODE)

With *m=1* and *n=0* , we get the dominant mode with the lowest cutoff frequency.

$$f_{c_{10}} = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]_{\substack{m=1\\n=0}} = \frac{\pi}{a}$$

$$f_{c_{10}} = \frac{1}{2\pi}\frac{1}{\sqrt{\varepsilon\mu}}\frac{\pi}{a} = \frac{1}{2a\sqrt{\varepsilon\mu}}$$

$$\lambda_{c_{10}} = \frac{2\pi}{k_{c_{10}}} = \frac{2\pi}{\pi/a} = 2a$$

$$\lambda_{g_{10}} = \frac{\lambda}{\sqrt{1-(\lambda/2a)^2}}$$

Note that the cutoff wavenumber, cutoff frequency, cut off wavelength all depends on the geometry.

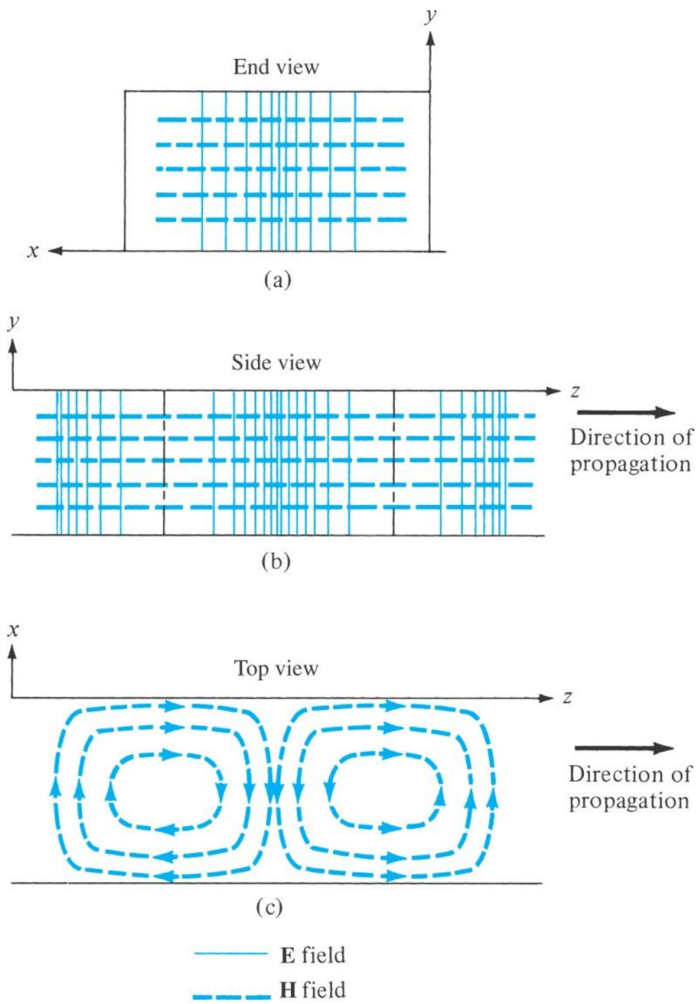
The Field Components of the Dominant Mode

$$H_z = A_{10}\cos\left(\frac{\pi x}{a}\right)e^{-j\beta_{10}z}$$
$$E_x = 0$$
$$E_y = \left(-\frac{j\omega\mu a}{\pi}\right)A_{10}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta_{10}z}$$
$$H_x = \frac{j\beta a}{\pi}A_{10}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta_{10}z}$$
$$E_z = 0$$
$$H_y = 0$$

With, $\beta = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}$

We see that the field components are independent of y. Only the y-directed component of the electric field exists and has a simple sinusoidal variation of strength across the width of the WG. The magnetic field exits in the closed loops in the x-z plane and the whole field pattern moves along the WG at the phase velocity $v_p = \frac{\omega}{\beta} = f\lambda_g$.

The electric field lines terminate in electric charges in the walls of the WG. As the wave travels along hte WG, the currents in the WG walls redistribute these charges so that the electric field always correctly terminated.



Field lines corresponding to TE_{10} MODE

<u>The Surface Currents on the walls of the WG (TE_{10} Mode)</u>

The surface current density on a perfect conductor is:

$$\bar{J}_s = \hat{n}\, X\, \bar{H}$$

<u>On x=0 wall</u>,

$$\hat{n} = \hat{a}_x, \quad \bar{J}_s = \hat{a}_x X \left(A_{10} e^{-j\beta_{10} z} \hat{a}_z \right)$$
$$\bar{J}_s = -\hat{a}_y A_{10} e^{-j\beta_{10} z}$$

On x=a wall,

$$\hat{n} = -\hat{a}_x, \quad \bar{J}_s = -\hat{a}_x X \left(-A_{10} e^{-j\beta_{10} z} \hat{a}_z \right)$$
$$\bar{J}_s = -\hat{a}_y A_{10} e^{-j\beta_{10} z}$$

On y=0 wall,

$$\hat{n} = \hat{a}_y, \quad \bar{J}_s = \hat{a}_y X \left(\frac{j\beta a}{\pi} A_{10} \sin\left(\frac{\pi x}{a}\right) \hat{a}_x + A_{10} \cos\left(\frac{\pi x}{a}\right) \hat{a}_z \right) e^{-j\beta_{10} z}$$
$$\bar{J}_s = A_{10} \left(\cos\left(\frac{\pi x}{a}\right) \hat{a}_x - \frac{j\beta a}{\pi} \sin\left(\frac{\pi x}{a}\right) \hat{a}_z \right) e^{-j\beta_{10} z}$$

On y=b wall,

$$\hat{n} = -\hat{a}_y, \quad \bar{J}_s = -\hat{a}_y X \left(\frac{j\beta a}{\pi} A_{10} \sin\left(\frac{\pi x}{a}\right) \hat{a}_x + A_{10} \cos\left(\frac{\pi x}{a}\right) \hat{a}_z \right) e^{-j\beta_{10} z}$$
$$\bar{J}_s = -A_{10} \left(\cos\left(\frac{\pi x}{a}\right) \hat{a}_x - \frac{j\beta a}{\pi} \sin\left(\frac{\pi x}{a}\right) \hat{a}_z \right) e^{-j\beta_{10} z}$$

The surface charge density ρ_s can be determined from $\rho_s = \hat{n}.\bar{D}$,

On x=0 wall,

$$\hat{n} = \hat{a}_x, \ \hat{n}.\bar{D} = \hat{a}_x.\left(\varepsilon_0 E_y \hat{a}_y\right) = 0$$
$$\rho_s = 0$$

On x=a wall,

$$\hat{n} = -\hat{a}_x, \ \hat{n}.\bar{D} = -\hat{a}_x.\left(\varepsilon_0 E_y \hat{a}_y\right) = 0$$
$$\rho_s = 0$$

On y=0 wall,

$$\hat{n}=\hat{a}_y,\quad \hat{n}.\bar{D}=\varepsilon_0 E_y\Big|_{y=0}=\frac{j\varepsilon_0\omega\mu a}{\pi}A_{10}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

$$\rho_s=-\frac{j\omega\varepsilon_0\mu a}{\pi}A_{10}\sin\frac{\pi x}{a}e^{-j\beta_{10}z}$$

On y=b wall,

$$\hat{n}=-\hat{a}_y,\quad \hat{n}.\bar{D}=\varepsilon_0 E_y\Big|_{y=0}=\frac{j\varepsilon_0\omega\mu a}{\pi}A_{10}\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

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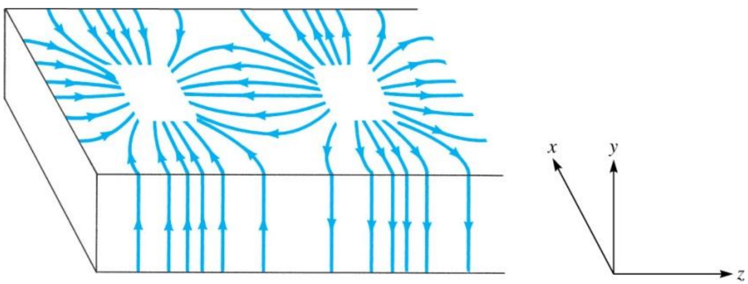
$$\rho_s=\frac{j\omega\varepsilon_0\mu a}{\pi}A_{10}\sin\frac{\pi x}{a}e^{-j\beta_{10}z}$$



Surface current on waveguide walls for TE_{10} mode.

The current is redistributing the charge in order to support the electric field intensity one quarter of a cycle later.

Power Flow down the Waveguide (TE_{10} Mode)

The time- average power per unit area is:

$$\bar{P}_{av}=\frac{1}{2}\mathrm{Re}\left(\bar{E}X\bar{H}^*\right)$$

$$\bar{E}X\bar{H}^*=\left(-\frac{j\omega\mu a}{\pi}A_{10}\sin\frac{\pi x}{a}e^{-j\beta_{10}z}\hat{a}_y\right)X\left[-\frac{j\beta a}{\pi}A_{10}^*\sin\left(\frac{\pi x}{a}\right)\hat{a}_x+A_{10}^*cos\left(\frac{\pi x}{a}\right)\hat{a}_z\right]e^{j\beta_{10}z}$$

$$\left(\bar{E}X\bar{H}^*\right).\hat{a}_z=-\frac{j\omega\mu a}{\pi}\left(-\frac{j\beta a}{\pi}\right)\sin^2\left(\frac{\pi x}{a}\right)\left|A_{10}\right|^2(-\hat{a}_z).\hat{a}_z$$

$$\left(\bar{E}X\bar{H}^*\right).\hat{a}_z=\frac{\omega\mu\beta a^2}{\pi^2}\left|A_{10}\right|^2\sin^2\left(\frac{\pi x}{a}\right)$$

For a propagating mode β is real, so

$$\bar{P}_{av}.\hat{a}_z = \frac{\omega\mu\beta a^2}{2\pi^2}|A_{10}|^2 \sin^2\left(\frac{\pi x}{a}\right)$$

The power crossing the WG cross-section:

$$P_{10,av} = \frac{\omega\mu\beta a^2}{2\pi^2}|A_{10}|^2 b\frac{a}{2} = \frac{\omega\mu\beta a^3 b}{4\pi^2}|A_{10}|^2$$

$$P_{10,av} = \frac{\omega\mu\beta a^3 b}{4\pi^2}|A_{10}|^2$$

<u>Characteristic Impedance of a Waveguide</u>

In order to apply the transmission line theory to WG's, we must be able to determine β or λ_g as well as the characteristic impedance Z_0 of the particular guide configuration. We have, $\beta = \frac{2\pi}{\lambda_g}$ and

$$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\left(f_c/f\right)^2\right)}}.$$

Define:

$Z_0 = \frac{V^+}{I^+}$,where V^+ is the voltage between the conductors and I^+ is the conduction current in the propagation direction for the traveling wave. For TEM lines Z_0 is uniquely defined since the value of $V^+ = -\int \bar{E}.d\bar{l}$ is independent of the integration path. In the WG case, V^+ is a function of the integration path.

For the dominant TE_{10} mode

$$Z_0 = 60\sqrt{\frac{\mu_r}{\varepsilon_r}}ab\left[1-\left(f_c/f\right)^2\right]^{-1/2}$$

ATTENUATION in a WAVEGUIDE

Attenuation in a WG can be caused by either dielectric loss and/or conductor loss. If α_d is the attenuation constant due to dielectric loss and α_c is the attenuation constant due to the conductor loss, then the total attenuation constant is:

$$\alpha = \alpha_c + \alpha_d$$

If the WG is completely filled with a homogeneous dielectric the attenuation can be calculated from the propagation constant and this result applies to any guide with a homogeneous dielectric filling the guide. $\gamma = \sqrt{k_c^2 - k^{*2}}$

k^*: Complex wavenumber. $\gamma = k_c^2 - \omega^2\varepsilon_0\mu_0\varepsilon_r\left(1 - j\tan\delta\right)^{1/2}$.

In practical applications. So, with $\sqrt{a^2+x^2} \simeq a + \frac{1}{2}\left(\frac{x^2}{a}\right)$ for $x << a$

Then, $\gamma = k_c^2 - \omega^2\varepsilon_0\mu_0\varepsilon_r\left(1 - j\tan\delta\right)^{1/2} \simeq \sqrt{k_c^2 - k^2} + \frac{1}{2}\frac{jk^2\tan\delta}{\sqrt{k_c^2-k^2}} = \frac{k^2\tan\delta}{2\beta} + j\beta$

Since $\sqrt{k_c^2 - k^2} = j\beta,\ k = \omega\sqrt{\varepsilon_0\varepsilon_r\mu_0}$

$$\gamma \simeq \frac{k^2\tan\delta}{2\beta} + j\beta$$

Then, $\alpha_d = \frac{k^2\tan\delta}{2\beta}(Np/m)$ For TE and TM waves.

For TEM waves, $k_c = 0,\ \beta = k$, $\alpha_d = \frac{k\tan\delta}{2}(Np/m)$

PERTURBATION METHOD FOR CALCULATING ATTENUATION DUE TO CONDUCTOR LOSS

The method uses the fields of the lossess line, with the assumption that the fields of the lossy line are not greatly different from the fields of the lossless line.

The power flow along a lossy TL, in the absence of reflections:

$$P(z) = P_0 e^{-2\alpha z}$$

Where P_0 is the power at z=0 plane and α is the attenuation constant to be determined. Now, define the power loss per unit length along the line as:

$$P_l(z) = -\frac{\partial P}{\partial z} = 2\alpha P_0 e^{-2\alpha z} = 2\alpha P(z)$$

Where the (-) sign on the derivative was chosen so that P_l would be a positive quantity. Then,

$$\alpha = \frac{P_l(z)}{2P(z)} = \frac{P_l(z)}{2P_0} e^{2\alpha z}$$

The power loss per unit length due to finite wall conductivity is:

$$P_l = \frac{R_s}{2} \oint_c |\bar{J}_s|^2 dl$$

R_s=the wall surface resistance.

C=integration contour which encloses the perimeter of the guide walls.

TM MODES

We have $H_z = 0$ and $e_z(x,y)$ must satisfy $\left(\frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_z}{\partial y^2} + k_c^2 e_z\right) = 0$, $\qquad k_c^2 = k^2 - \beta^2$

The general solution is,

$$e_z(x,y) = (A\cos k_x x + B\sin k_x x)(C\cos k_y y + D\sin k_y y)$$

The boundary conditions are:

$e_z(x,y)=0$ for $x=0$ and $x=a$

$e_z(x,y)=0$ for $y=0$ and $y=b$

Using these B.C.'s we obtain:

A=0 and $k_x=\dfrac{m\pi}{a},\ \ m=1,2,3,\ldots$

C=0, and $k_y=\dfrac{n\pi}{b},\ \ n=1,2,3,\ldots$

So,

$$E_z(x,y,z)=B_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

The other field components are:

$$E_x(x,y,z)=-\frac{j\beta m\pi}{ak_c^2}B_{mn}cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$E_y(x,y,z)=-\frac{j\beta n\pi}{bk_c^2}B_{mn}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$H_x(x,y,z)=\frac{j\omega\varepsilon\pi n}{bk_c^2}B_{mn}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$H_y(x,y,z)=-\frac{j\omega\varepsilon\pi m}{ak_c^2}B_{mn}cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$\beta=\sqrt{k^2-k_c^2}=\left[k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

If either *m* or *n* is zero, the fields vanish identically. So there are no TM_{00}, TM_{01} or TM_{01} modes.

The lowest order TM mode is the TM_{11} mode with:

$$f_{c_{11}}=\frac{1}{2\pi\sqrt{\varepsilon\mu}}\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^{1/2}>f_{c_{TE10}}$$

Since f_c of the lowest order TM mode is greater than the f_c of the lowest order TE mode, TE_{10} is the lowest among all modes.

The wave-impedance

$$Z_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\eta}{k}\beta$$