Hyperbolic Functions

المحاضرة العاشرة
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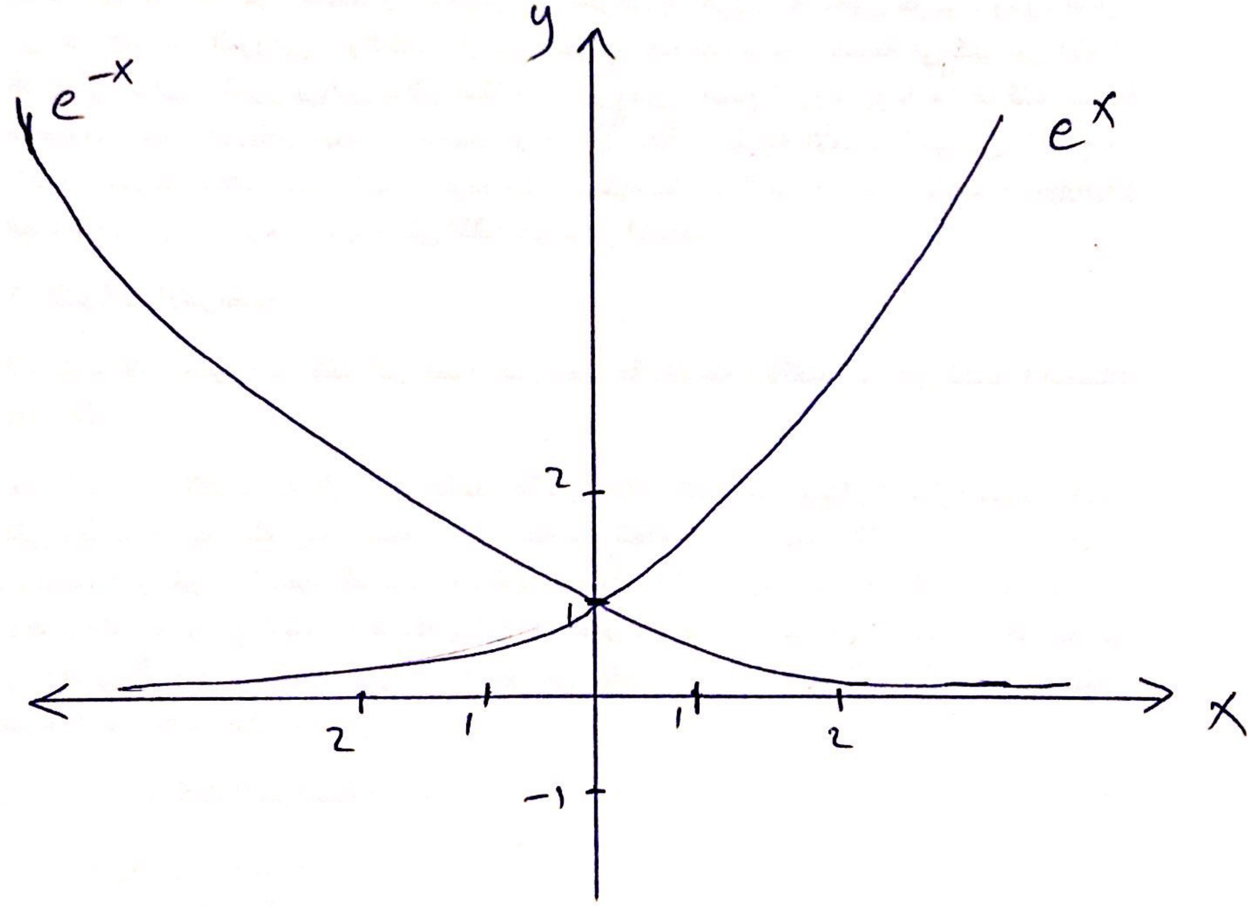
They describe the motion of waves in elastic solids, the shapes of hanging, electric power Lines, and temperature distributions.

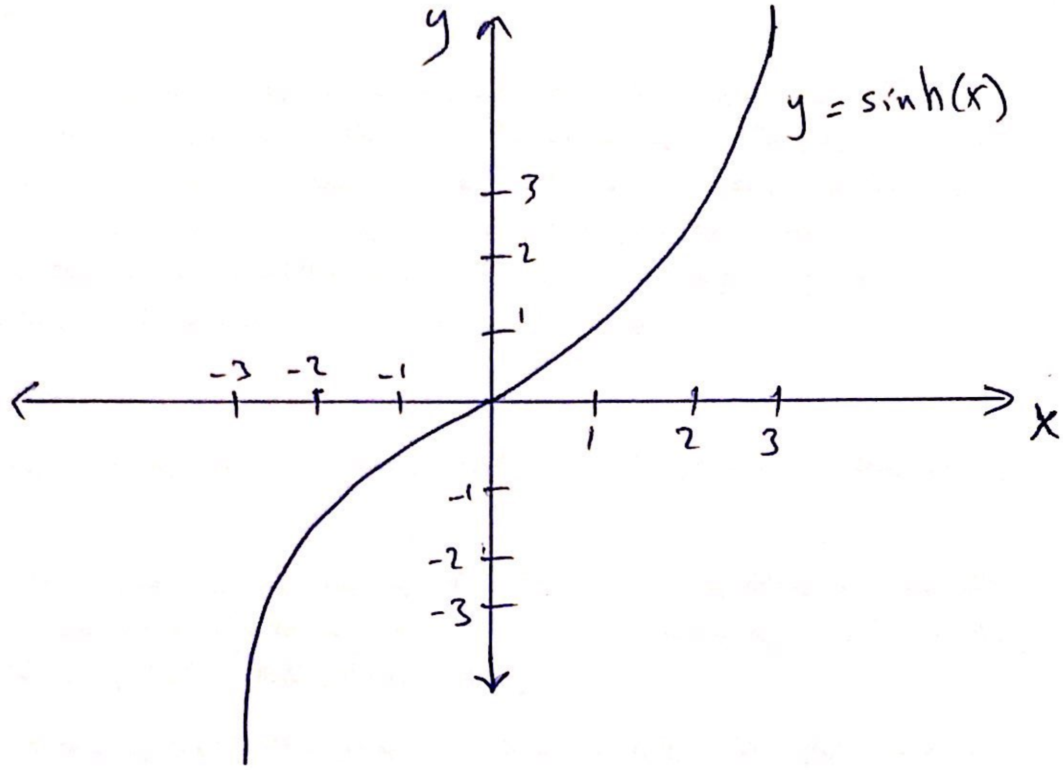


Note that:

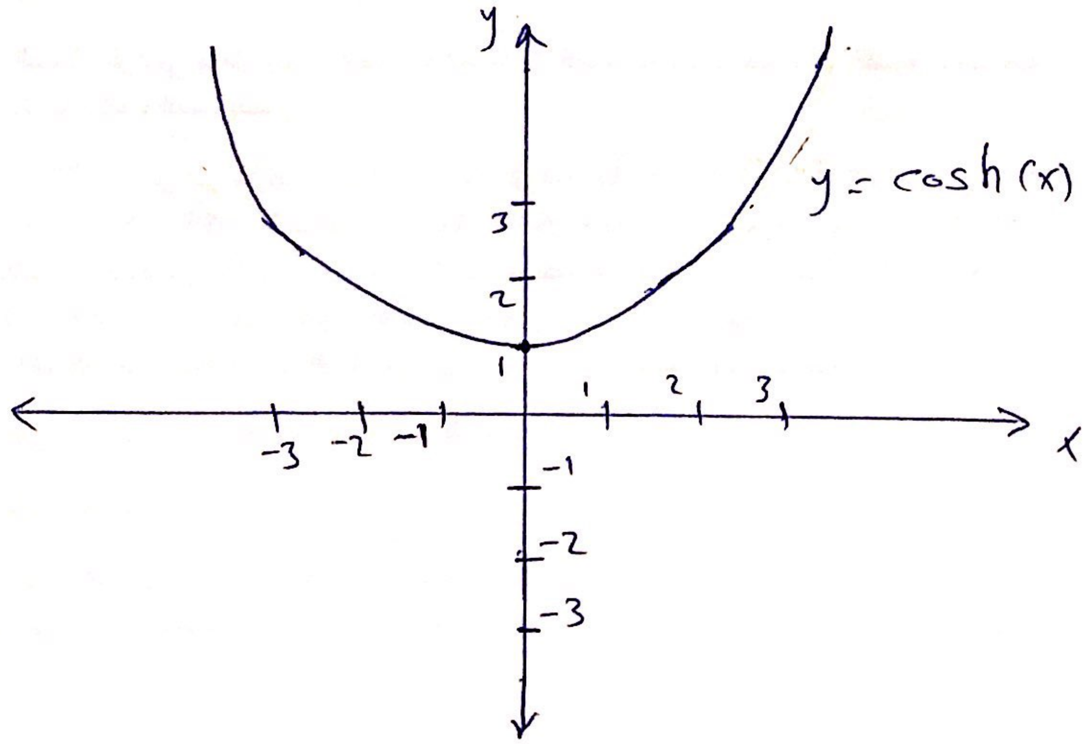
$$e^{x} = \underbrace{\frac{e^{x}+e^{-x}}{2}}_{\text{Even Part}} + \underbrace{\frac{e^{x}-e^{-x}}{2}}_{\text{Odd part}}$$

Hyperbolic cosine   Hyperbolic sine →

① Hyperbolic sine $\Rightarrow$ $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

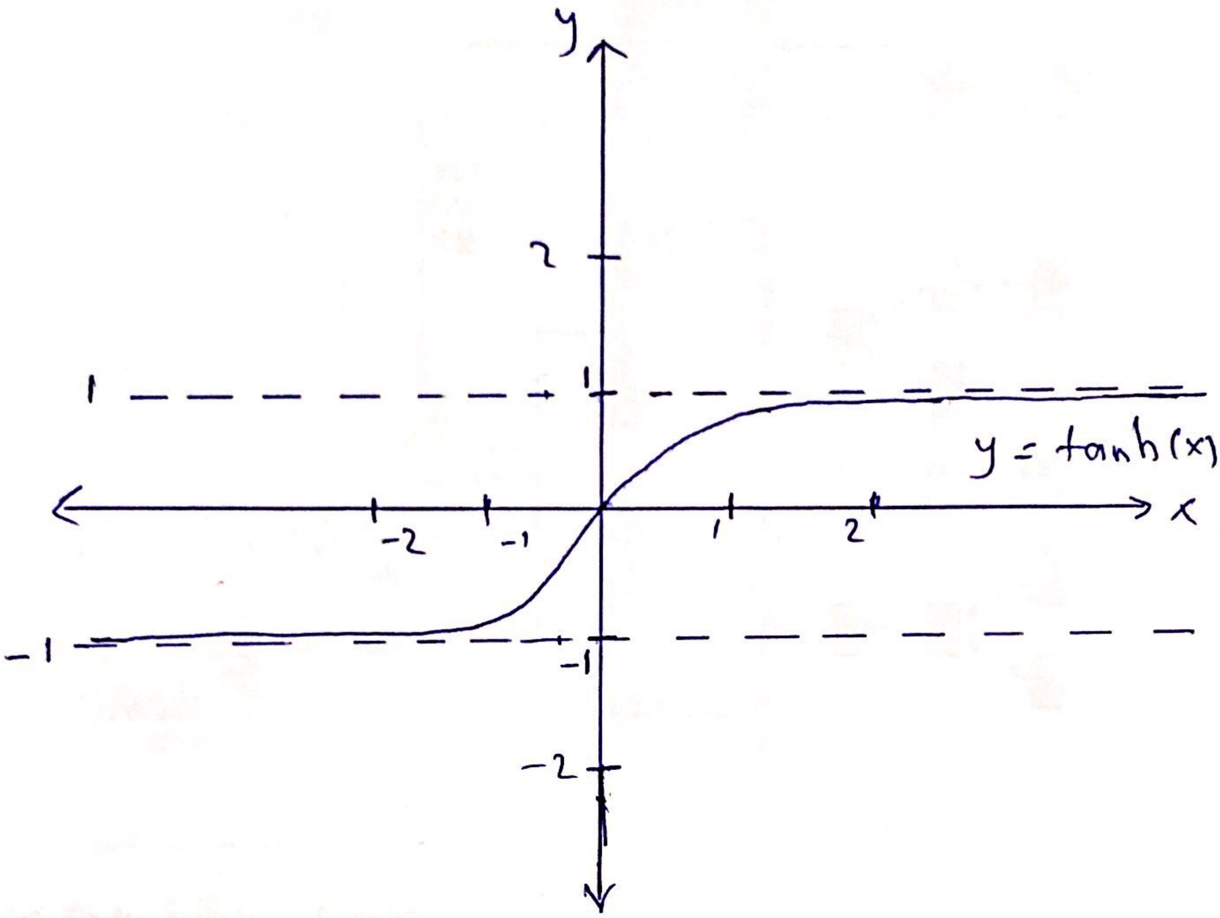


② Hyperbolic cosine $\Rightarrow$ $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$



(3) Hyperbolic tangent $\Rightarrow$ $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$

$$= \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$



(4) Hyperbolic cotangent $\Rightarrow$ $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)}$

$$= \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

(5) Heperbolic secant $\Rightarrow$ $\mathrm{sech}(x) = \dfrac{1}{\cosh(x)}$

$$= \frac{2}{e^{x} + e^{-x}}$$

$\longrightarrow$

6. Hyperbolic cosecant $\Rightarrow$ $\operatorname{csch}(x) = \dfrac{1}{\sinh(x)}$

$$= \frac{2}{e^{x} + e^{-x}}$$

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*Identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

$$\operatorname{csch}^2 x = -1 + \coth^2 x$$

$$\sinh(2x) = 2\sinh(x) \cdot \cosh(x)$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$$

$$\cosh^2(x) = \frac{1}{2}\left(\cosh(2x) + 1\right)$$

$$\sinh^2(x) = \frac{1}{2}\left(\cosh(2x) - 1\right)$$

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*Excersize: Prove that:

$$\cosh^2 x - \sinh^2 x = 1$$

solution:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}\left(\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}\right) - \frac{1}{4}\left(\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}\right)$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1 \quad \#$$

* Derivatives and Integrals:

There are similarities with trigonometric functions:

$$\frac{d}{dx}(\sinh(u)) = \cosh(u)\frac{du}{dx}$$

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$$\frac{d}{dx}(\tanh(u)) = \operatorname{sech}^2(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\coth(u)) = -\operatorname{csch}^2(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech}(u)) = -\operatorname{sech}(u) \cdot \tanh(u) \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch}(u)) = -\operatorname{csch}(u) \cdot \coth(u) \cdot \frac{du}{dx}$$

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* Excersize: Prove that $\frac{d}{dx}(\sinh(u)) = \cosh(u)\frac{du}{dx}$

solution: $\frac{d}{dx}(\sinh(u)) = \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right)$

$$= \frac{e^u + e^{-u}}{2} * \frac{du}{dx}$$

$$= \cosh(u) * du/dx$$

* The derivatives formulas produce the integral formulas :

$$\int \sinh(u)\, du = \cosh(u) + c$$

$$\int \cosh(u)\, du = \sinh(u) + c$$

$$\int \operatorname{sech}^2(u)\, du = \tanh(u) + c$$

$$\int \operatorname{csch}^2(u)\, du = -\coth(u) + c$$

$$\int \operatorname{sech}(u) \tanh(u)\, du = -\operatorname{sech}(u) + c$$

$$\int \operatorname{csch}(u) . \coth(u)\, du = -\operatorname{csch}(u) + c$$

Ex.1: $\int \coth(5x)\, dx$

$$= \frac{5}{5} \int \frac{\cosh(5x)}{\sinh(5x)}\, dx$$

$$= \frac{1}{5} \ln|\sinh(5x)| + c$$

⟹

Ex.2 : Prove that :

$$\int \sinh(u)\,du = \cosh(u) + c$$

Ex.3 : Find $\frac{d}{dx}\left(\cosh(u^{-3})\right)$

solution:

$$\frac{d}{dx}\left(\cosh(u^{-3})\right)$$

$$= \frac{d}{dx}\left(\frac{e^{u^{-3}} + e^{-u^{-3}}}{2}\right)$$

$$= \frac{-3u^{-4} e^{u^{-3}} + 3u^{-4} e^{u^{-3}}}{2} * \frac{du}{dx}$$

$$= -3u^{-4} * \frac{e^{u^{-3}} - e^{u^{-3}}}{2} * \frac{du}{dx}$$

$$= -3u^{-4} * \sinh(u^{-3}) * \frac{du}{dx}$$

Ex.4 : $\int_0^1 \sinh^2 x \, dx$

Solution :

$$\int_0^1 \sinh^2 x \, dx = \frac{1}{2}\int_0^1 (\cosh(2x) - 1)\, dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\sinh(2x) - x\right]_0^1$$

$$= \frac{1}{2}\left[\left(\frac{\sinh(2)}{2} - (1)\right) - (0)\right]$$

$$= \frac{1}{2} * 0.813$$

$$= 0.407$$

Ex.5: $\int_0^4 \frac{\cosh(\sqrt{x})}{\sqrt{x}} \, dx$

Ex.6: $\int_0^{\ln 2} 4e^x \sinh(x) \, dx$

Note: Hyperbolic Functions at section 7.8
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