# Matrices

المحاضرة الحادية عشر
المصفوفات

Any set of numbers which can be put at a specified arrangment, between two brackets is called a matrix.

القطر الرئيسي

- $$\begin{bmatrix} 5 & 3 & 1 \\ 2 & 0 & -2 \\ 1 & 4 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$$

  $3 * 3 \qquad 3 \times 2 \qquad 1 \times 3$

- Each matrix consists of a number of rows (M) and colomns (N), so the volume of the matrix is $M \times N$.

Each number in the matrix is called element, where its location can be specefied by the row and colomn index $a_{ij}$.

* Squar matrix: is the matrix where $M = N$.

* Zero matrix: is the matrix where all its elements equal to zero.

* Unit matrix (I): is the matrix where the elements at: $i = j$ are (1)s, and (0)s else where (identity matrix).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* Any two matrices A, B are said to be equal if and only if all the elements $a_{ij} = b_{ij}$.

*Operation on matrices

① Addition :

to add two matrices A & B the volume of those matrices should be the same .

$$2 \times 3 \quad + \quad 2 \times 3$$

$$3 \times 2 \quad + \quad 3 \times 2$$

Example: Find $A + B$ , $A - B$

if $A = \begin{bmatrix} 1 & 0 & 5 \\ 4 & 2 & 1 \\ -1 & 3 & -2 \end{bmatrix}$

and $B = \begin{bmatrix} 0 & 1 & 0 \\ 10 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$

→

solution:

$$A + B = \begin{bmatrix} 1 & 1 & 5 \\ 14 & 4 & -2 \\ -1 & 4 & -2 \end{bmatrix}$$

- $$A - B = \begin{bmatrix} 1 & -1 & 5 \\ -6 & 0 & 4 \\ -1 & 2 & -2 \end{bmatrix}$$

---

② Multiplication

① constant multiplication:

$k.A$ , $k$ : constant

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$k * A = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = k \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



$\Rightarrow$

## (ii) Matrices multiplication:

To multiply two matrices A and B, then the number of columns in A should equal to the number of rows in B.

Example: Find $A \times B$,

- where $A = \begin{bmatrix} 3 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \end{bmatrix}$

$\underline{\underline{3 \times 2}}$ $\quad\quad$ $\underline{\underline{2 \times 3}}$

solution:

- $A \times B = \begin{bmatrix} 8 & 4 & 10 \\ -2 & -1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$

## * Transpose of a matrix :

$$A[a_{ij}] \longrightarrow A^T[a_{ji}]$$

$$A = \begin{bmatrix} 3 & 1 & 5 \\ 6 & 2 & 1 \end{bmatrix} \qquad 2 \times 3$$

$$A^T = \begin{bmatrix} 3 & 6 \\ 1 & 2 \\ 5 & 1 \end{bmatrix} \qquad 3 \times 2$$

* Determinat :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = ad - bc$$

---

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ek - fh) - b(dk - fg) + c(dh - eg)$$

Example: if

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$ Find Det. A

solution:

$$|A| = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} - 3 \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 5 - 3(-2) + (-1)$$

$$= 10$$

## $\alpha$ Matrix :

It is a matrix consisting of elements which are replacement to another matrix of the same size.

$$\alpha_{ij} = (-1)^{i+j} |B|$$

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

$$\alpha_{1,1} = (-1)^{1+1} |4| = 4$$

$$\alpha_{1,2} = (-1)^{1+2} |3| = -3$$

$$\alpha_{2,1} = (-1)^{2+1} |1| = -1$$

$$\alpha_{2,2} = (-1)^{2+2} |2| = 2$$

$$\alpha = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

* Example:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

find $\alpha$

solution:

$$\alpha_{1,1} = (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2$$

$$\alpha_{1,2} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = 4$$

$$\alpha_{1,3} = (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = -2$$

$$\alpha_{2,1} = (-1)^{2+1} \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} = +1$$

$$\alpha_{2,2} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2$$

$$\alpha_{2,3} = (-1)^{2+3} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5$$

$$\alpha_{3,1} = (-1)^{3+1} \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5$$

$$\alpha_{3,2} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2$$

$$\alpha_{3,3} = (-1)^{3+3} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore \quad \alpha = \begin{bmatrix} -2 & 4 & -2 \\ 1 & -2 & 5 \\ 5 & -2 & 1 \end{bmatrix}$$

* Adjoint Matrix

$$adj\ A = [\alpha^T]$$

for the previous example:

$$adj\ A = \begin{bmatrix} -2 & 1 & 5 \\ 4 & -2 & -2 \\ -2 & 5 & 1 \end{bmatrix}$$

---

* Matrix inversion:

If $[A] \times [B] = [I]$, then it can be said that $[B]$ is the inverse of $[A]$ or it is $[A^{-1}]$.

$$[A^{-1}] = \frac{Adj\ A}{|A|}$$

→

for the previous example:

$$|A| = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= -2 - 3(-4) + (-2)$$

$$= -2 + 12 - 2$$

$$= 8$$

$$\therefore [A^{-1}] = \frac{1}{8}\begin{bmatrix} -2 & 1 & 5 \\ 4 & -2 & -2 \\ -2 & 5 & 1 \end{bmatrix}$$

$$[A^{-1}] = \begin{bmatrix} \frac{-1}{4} & \frac{1}{8} & \frac{5}{8} \\ \frac{1}{2} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{5}{8} & \frac{1}{8} \end{bmatrix}$$

→

للتأكد من صحة الحل:

$$[A] \times [A^{-1}] = [I]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

unit or identity matrix

---

*Note:

Signular Matrix: A squar matrix that does not has an inverse. i.e. has zero determinate.

* Solving a System of linear equations with matrices :

① Grammer 's Rule :

$$[a]\,[x] = [b]$$

$[a]$ → مصفوفة الثوابت

$[x]$ → مصفوفة المتغيرات

$[b]$ → مصفوفة النواتج

ملاحظة: اذا عدد المعادلات = أقل من عدد المجاهيل ، نعطي عدد لا نهائي من الحلول .

$$x_1 = \frac{|x_1|}{|a|}$$

$$x_2 = \frac{|x_2|}{|a|}$$

*Example ①: Solve the following Linear System.

$$2x + 3y = 5$$
$$3x - y = 2$$

* Solution:

$$\underset{2\times 2}{\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}$$

$$|a| = (2*-1) - (3*3) = -11$$

$$X = \frac{\begin{vmatrix} 5 & 3 \\ 2 & -1 \end{vmatrix}}{-11} = \frac{-11}{-11} = 1$$

للتأكد من صحة الحل
نعوض بالمعادلات

$$y = \frac{\begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix}}{-11} = \frac{-11}{-11} = 1$$

---

*Example ②: Solve the following Linear System by using matrices:

$$5x + 3y - z = 6$$

$$2x - y + z = 3$$

$$x + y + 2z = 6$$

Home Work

Answer:

$x = 1$ , $y = 1$ , $z = 2$

## ② Matrix inversion method:

$$[a][x] = [b]$$

مصفوفة الثوابت — مصفوفة المتغيرات — مصفوفة النواتج

$$[x] = [a^{-1}][b]$$

Example: Solve the following Linear system by use matrices.

$$2x + y = 4$$

$$3x + y + z = 6$$

$$y + z = 3$$

solution:

$$[a][x] = [b]$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$$

$\Longrightarrow$

$$[x] = [a^{-1}][b]$$

we need to find $[a^{-1}] = \dfrac{adj\,[a]}{|a|}$,

$$\alpha_{1,1} = (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$\alpha_{1,2} = (-1)^{3} \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = -3$$

$$\alpha_{1,3} = (-1)^{4} \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = 3$$

$$\alpha_{2,1} = (-1)^{3} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

$$\alpha_{2,2} = (-1)^{4} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2$$

$$\alpha_{2,3} = (-1)^{5} \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2$$

$$\alpha_{3,1} = (-1)^{4} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = +1$$



$\Rightarrow$

$$\alpha_{3,2} = (-1)^5 \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = -2$$

$$\alpha_{3,3} = (-1)^6 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1$$

$$\alpha = \begin{bmatrix} 0 & -3 & 3 \\ -1 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}$$

$$Adj[a] = [\alpha^T]$$

$$Adj[a] = \begin{bmatrix} 0 & -1 & 1 \\ -3 & 2 & -2 \\ 3 & -2 & -1 \end{bmatrix}$$

$$|a| = 0 - \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} + 0$$

$$= -3$$

$\Rightarrow$

$$[a^{-1}] = \frac{-1}{3}\begin{bmatrix} 0 & -1 & 1 \\ -3 & 2 & -2 \\ 3 & -2 & -1 \end{bmatrix}$$

$$[a^{-1}] = \begin{bmatrix} 0 & \frac{1}{3} & \frac{-1}{3} \\ 1 & \frac{-2}{3} & \frac{2}{3} \\ -1 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$\therefore$ According to matrix inversion method:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{3}\underset{3\times 3}{\begin{bmatrix} 0 & -1 & 1 \\ -3 & 2 & -2 \\ 3 & -2 & -1 \end{bmatrix}}\underset{3\times 1}{\begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}}$$

$$= \frac{-1}{3}\begin{bmatrix} -3 \\ (-12+12-6) \\ (12-12-3) \end{bmatrix}$$

$$= \frac{-1}{3}\begin{bmatrix} -3 \\ -6 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ $x = 1$,

$y = 2$,

$z = 1$

Example ②: Solve the following equation:

$$3X + A - I = 2X - B$$

where

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & -1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$I$ is identity matrix

solution:

$$\underline{X} + A - I = -B$$

الحل

$\Longrightarrow$

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -\begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$\therefore$ $x_1 + 1 - 1 = -3 \Rightarrow x_1 = -3$

$x_2 + 0 - 0 = -1 \Rightarrow x_2 = -1$

$x_3 + 2 - 0 = -1 \Rightarrow x_3 = -3$

$y_1 + 3 - 0 = -1 \Rightarrow y_1 = -4$

$\vdots$