* Limits — المحاضرة الثالثة

* In this lecture we will use limits to describe the way the function f varies.

* Some functions vary continnosly, small changes in x produce only small changes in $f(x)$. Other functions can have values that jump or vary erratically.

* The notion of limit gives a precise way to distinguish between these behaviors.

* If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to $\underline{\underline{a}}$ (but not equal to $\underline{\underline{a}}$) then we write:

$$\lim_{x \to a} f(x) = L$$

$$\lim_{x \to a} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches $\underline{a}$ is L".

* Some basic Limits

Let a and k be real numbers:

$$\lim_{x \to a} k = k \quad , \quad \lim_{x \to a} x = a$$

* Example:

$$\lim_{x \to 25} 3 = 3 \quad , \quad \lim_{x \to 0} 3 = 3 \quad , \quad \lim_{x \to \pi} 3 = 3$$

* Properties of Limits:

Let a be a real number and suppose that:

$$\lim_{x \to a} f(x) = L \quad , \quad \lim_{x \to a} g(x) = M$$

then:

a.) $$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

b.) $$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

c.) $$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L * M$$

d.) $$\lim_{x \to a} [f(x)/g(x)] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$$ provided $M \neq 0$

e.) $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$

$= \sqrt[n]{L}$ provided $L \geq 0$

Example: Find $\lim\limits_{x \to 5} (x^2 - 4x + 3)$

$= 25 - 20 + 3$

$= 8$

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*Right side Limit and Left Side Limit

Definition: A function $f(x)$ has a limit as x approaches $\underline{a}$ if and only if the Right side and the Left side Limits at $\underline{a}$ exist and are equal.

$$\lim_{x \to a} f(x) = L \overset{\text{iff}}{\Longleftrightarrow} \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

$\rightarrow$

*Example: Let

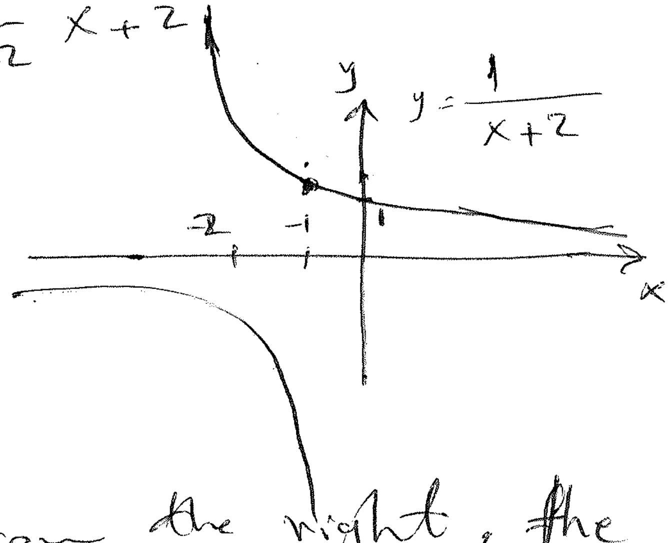
$$f(x)=\begin{cases}1/(x+2) & x<-2\\ x^2-5 & -2<x\leq 3\\ \sqrt{x+13} & x>3\end{cases}$$

Find (a) $\lim_{x\to -2} f(x)$, (b) $\lim_{x\to 0} f(x)$, (c) $\lim_{x\to 3} f(x)$

solution:

(a) As x approaches -2 from the left, the formula for f is $f(x)=\frac{1}{x+2}$ so that:

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{1}{x+2} = -\infty$$



As x approaches -2 from the right, the formula for f is $f(x)=x^2-5$, so that:

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (x^2-5) = (-2)^2-5 = -1$$

Since $\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$ $\boxed{\therefore \lim_{x\to -2} f(x) \text{ does not Exist}}$

(b) $\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 - 5)$

$= -5$

(c) $\lim_{x \to 3} f(x)$

As x approaches 3 from the left, the formula for f is $f(x) = (x^2 - 5)$, so that:

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 5)$

$= 9 - 5$

$= 4$

As x approaches 3 from the righ, the formula for f is $f(x) = \sqrt{x + 13}$, so that:

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \sqrt{x + 13}$

$= \sqrt{16} = 4$

$\therefore \lim_{x \to 3} f(x) = 4$ since $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$

* Limits at infinity :

Limit as $x \to \infty$ or $x \to -\infty$

Example :

$f(x) = 3$ then

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 3 = 3$$

Example :

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

* Properties of finite Limits as $x \to \pm\infty$ are the same properties of Limits as $x \to a$

$\rightarrow$

* Limits of rational function as $x \to \pm\infty$

Consider the rational function as $f(x) = \frac{P(x)}{q(x)}$

where $P(x)$ and $q(x)$ are polynomials.

ⓐ If the degree of $P(x) > q(x)$ then

$$\lim_{x \to \infty} f(x) = \infty \quad \text{or} \quad -\infty$$

ⓑ If the degree of $q(x) > P(x)$ then

$$\lim_{x \to \infty} f(x) = 0$$

ⓒ If the degree of $q(x) = P(x)$ then

$\lim_{x \to \infty} f(x)$ is a finite number

→

* Example :

$$\lim_{x \to \infty} \frac{-4x^3 + 7x}{2x^2 - 3x + 10}$$

$p(x) > q(x) \begin{cases} \infty \\ -\infty \end{cases}$

solution :

$$= \lim_{x \to \infty} \frac{-\frac{4x^3}{x^2} + \frac{7x}{x^2}}{2\frac{x^2}{x^2} - \frac{3x}{x^2} + \frac{10}{x^2}}$$

$$= \lim_{x \to \infty} \frac{-4x + \frac{7}{x}}{2 - \frac{3}{x} + \frac{10}{x^2}}$$

$$= \frac{-4(\infty) + 0}{2 - 0 + 0}$$

$$= \frac{-\infty}{2}$$

$$= -\infty$$

* Example

$p(x) = q(x) \rightarrow$ Finite number

$$\lim_{x \to \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

$$= \lim_{x \to \infty} \frac{5\frac{x^2}{x^2} + \frac{8x}{x^2} - \frac{3}{x^2}}{3\frac{x^2}{x^2} + \frac{2}{x^2}}$$

$$= \lim_{x \to \infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}}$$

$$= \frac{5 + \frac{8}{\infty} - \frac{3}{\infty}}{3 + \frac{2}{\infty}}$$

$$= \frac{5 + 0 - 0}{3 + 0}$$

$$= \frac{5}{3}$$

$y = \frac{5}{3}$

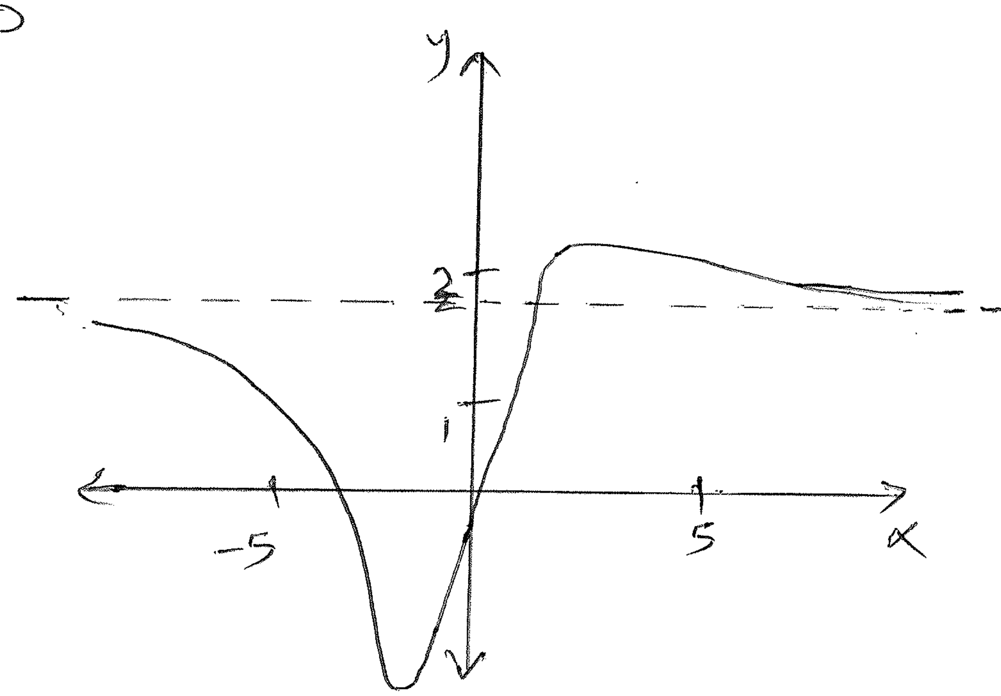


Fig. 2.33

* Example

$$\lim_{x \to \infty} \frac{11x + 2}{3x^3 - 1} \qquad p(x) < q(x)$$

$$= \lim_{x \to \infty} \frac{\frac{11x}{x^3} + \frac{2}{x^3}}{\frac{3x^3}{x^3} - \frac{1}{x^3}}$$

$$= \lim_{x \to \infty} \frac{\frac{11}{x^2} + \frac{2}{x^3}}{3 - \frac{1}{x^3}}$$

$$= \frac{0 + 0}{3 - 0}$$

$$= \frac{0}{3}$$

$$= 0$$

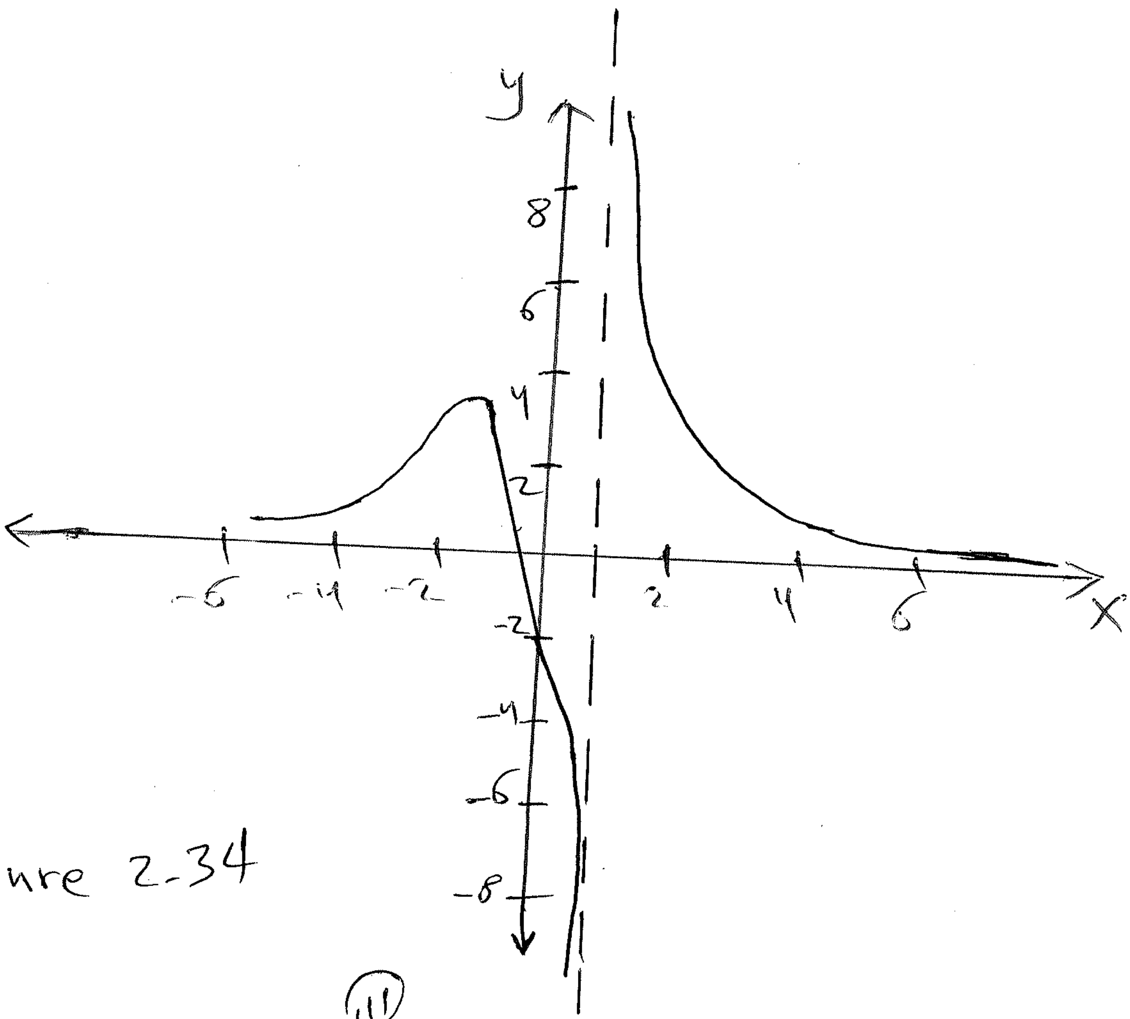


Figure 2-34

Example :

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \frac{0}{0}$$ نبحث عن النهاية بطرق أخرى

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (x+1)$$

$$= 2$$

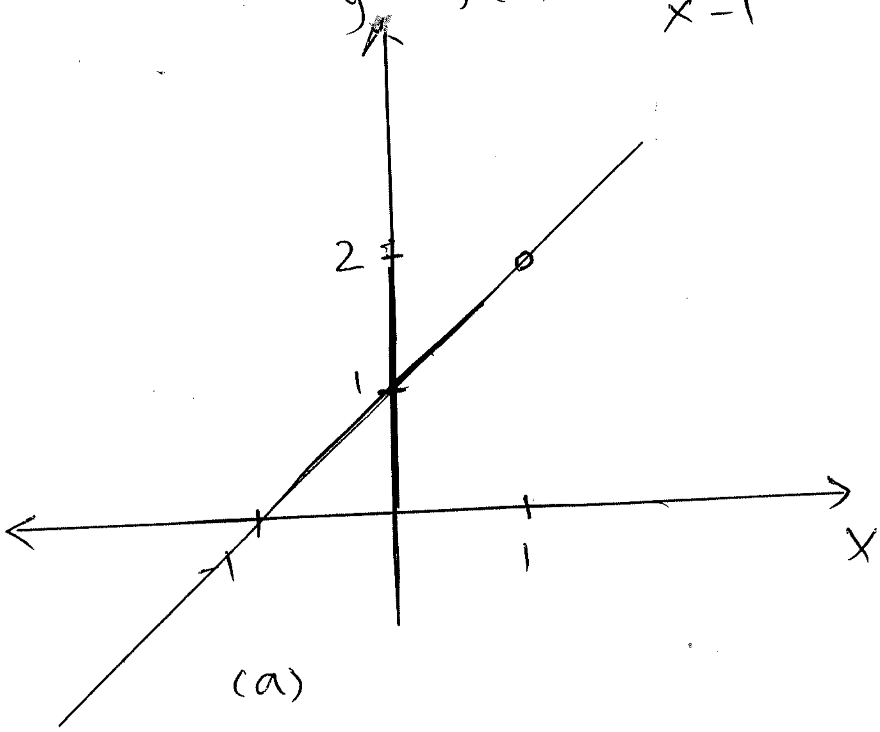
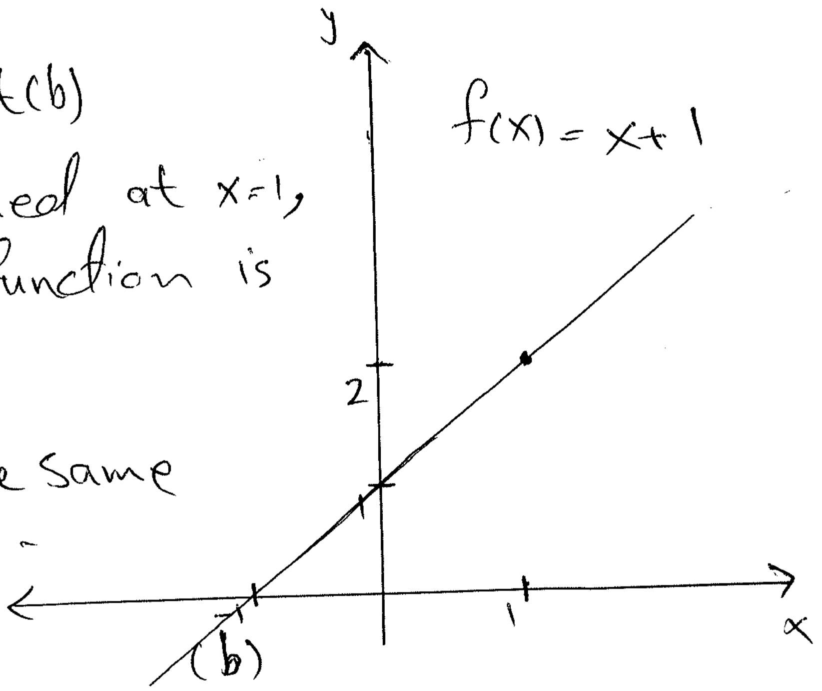


Figure 2.5:
The graph of $f(x) = \frac{x^2-1}{x-1}$ in part (a) is the same as the graph of $f(x) = x+1$ in part (b) except in part (b) the function is defined at $x=1$, while in part (a) the function is undefined at $x=1$.
The functions have the same Limits as $x \to 1$.



Example

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$$

solution

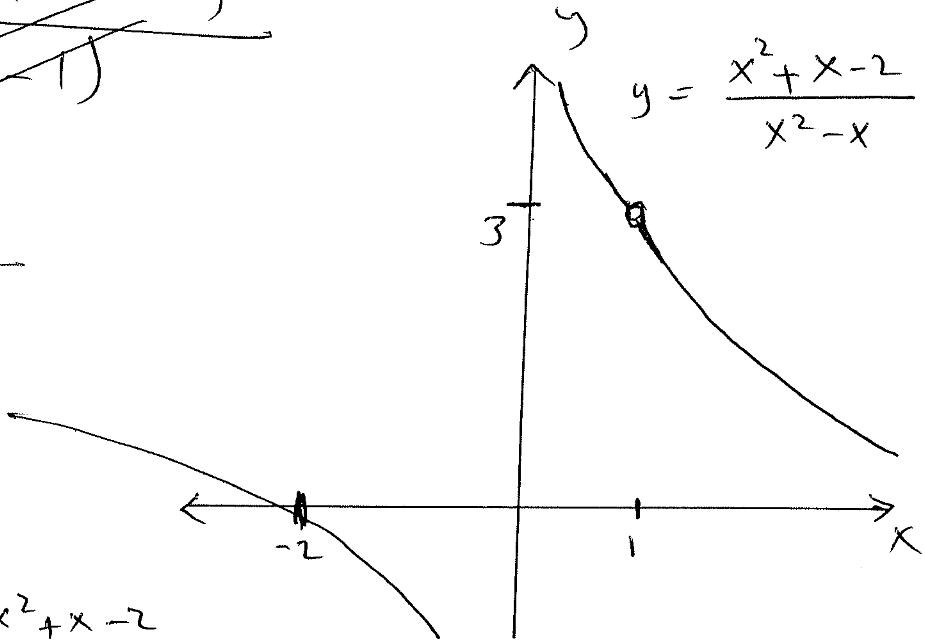
$$= \frac{(1)^2 + 1 - 2}{(1)^2 - 1}$$

$$= \frac{0}{0}$$ نبحث عن النهاية بطرق اخرى

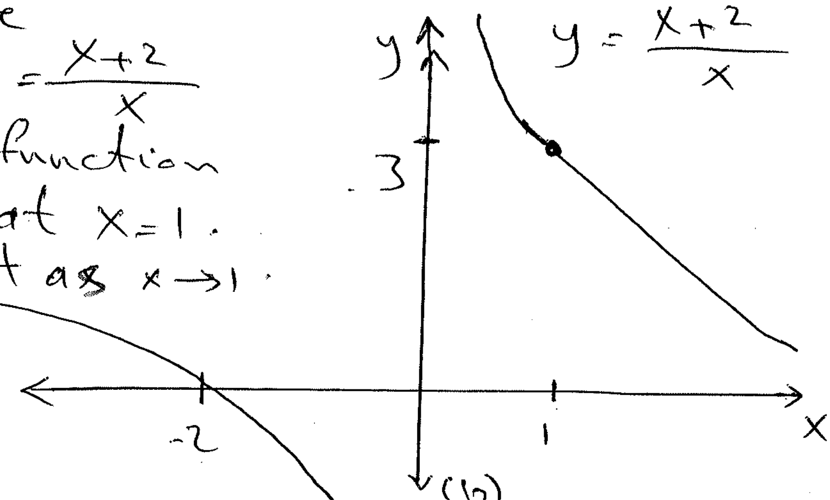
$$= \lim_{x \to 1} \frac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}}$$

$$= \lim_{x \to 1} \frac{x+2}{x}$$

$$= 3$$



(a)



(b)

Fig. 2.8: The graph of $f(x) = \frac{x^2+x-2}{x^2-x}$ in part (a) is the same as the graph of $f(x) = \frac{x+2}{x}$ in part (b) except the function in part (b) is defined at $x=1$. Both have the same limit as $x \to 1$.

Example :

$$\lim_{x \to 0} \frac{\sqrt{x^2+100}-10}{x^2}$$

Solution :

$$= \frac{10-10}{0} = \frac{0}{0}$$ نبحث عن النهاية بطرق اخرى

$$\lim_{x \to 0} \frac{\sqrt{x^2+100}-10}{x^2} * \left(\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}\right)^{\to 1}$$

$$= \lim_{x \to 0} \frac{\cancel{x^2}+\cancel{100}-\cancel{100}}{\cancel{x^2}\left(\sqrt{x^2+100}+10\right)}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{10+10}$$

$$= \frac{1}{20}$$

*Example: $\lim\limits_{x \to 2} \dfrac{2-x}{(x-2)^3}$

solution

$= \dfrac{2-2}{(2-2)^3} = \dfrac{0}{0}$ نبحث عن النهاية بطرق اخرى

$= \lim\limits_{x \to 2} \dfrac{2-x}{(x-2)^3}$

$= \lim\limits_{x \to 2} \dfrac{-\cancel{(x-2)}}{\cancel{(x-2)}^3}$

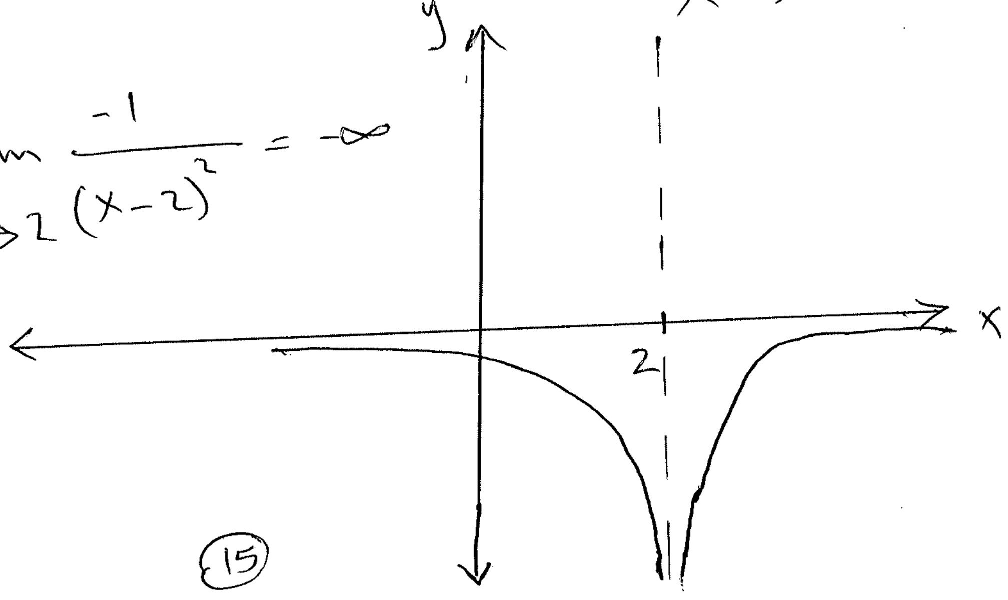
$= \lim\limits_{x \to 2} \dfrac{-1}{(x-2)^2}$

لا يمكن التبسيط أكثر

لذلك نستعين بالرسم حتى نعرف كيف تتصرف

هذه الدالة عند $x \to 2$

$\therefore \lim\limits_{x \to 2} \dfrac{-1}{(x-2)^2} = -\infty$



* Example :

① $\lim\limits_{x \to -2} \dfrac{x+3}{x+2}$, ② $\lim\limits_{x \to \infty} \dfrac{x+3}{x+2}$, ③ $\lim\limits_{x \to -\infty} \dfrac{x+3}{x+2}$

solution :

② $\lim\limits_{x \to \infty} \dfrac{x+3}{x+2}$ $\qquad p(x) = q(x)$

$$= \lim_{x \to \infty} \frac{\frac{x}{x} + \frac{3}{x}}{\frac{x}{x} + \frac{2}{x}}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{3}{x}}{1 + \frac{2}{x}}$$

$$= \cancel{\lim_{x \to \infty}} \frac{1 + \frac{3}{\infty}}{1 + \frac{2}{\infty}}$$

$$= \cancel{\lim_{x \to}} \frac{1 + 0}{1 + 0}$$

$$= 1$$

$$\lim_{x \to -2} \frac{x+3}{x+2} \longrightarrow$$

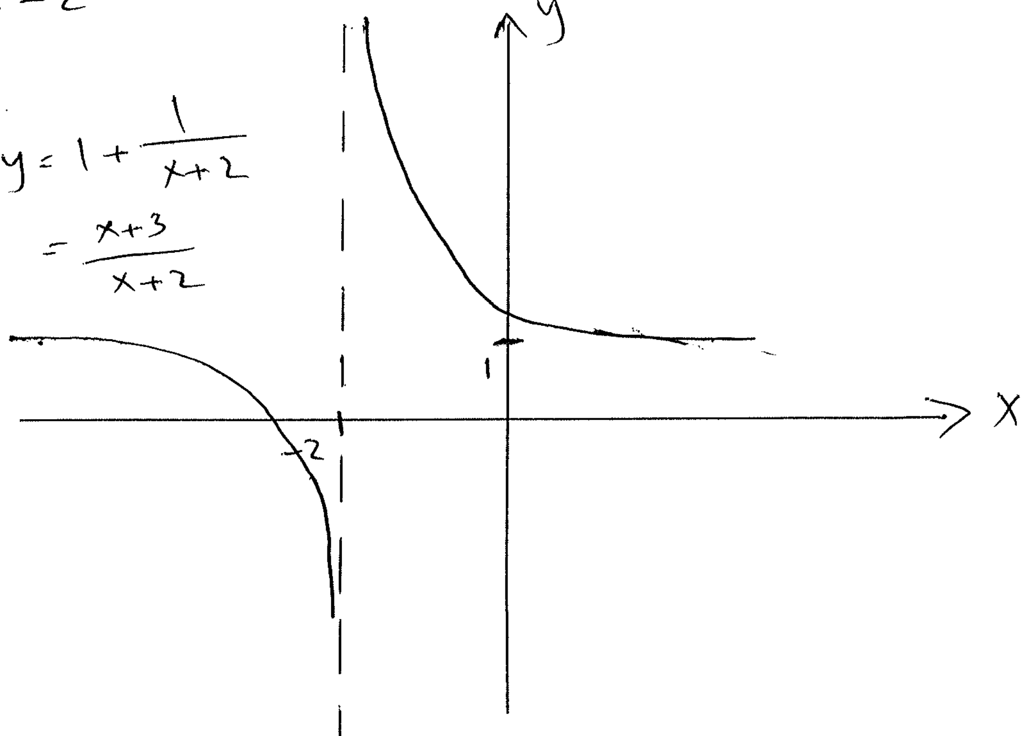
solution

$$= \frac{-2+3}{0} = \frac{+1}{0}$$ Undefined Value

$$\lim_{x \to -2} \frac{x+3}{x+2} \Longrightarrow$$ من الممكن تبسيط الكسر باجراء قسمة طويلة

$$\begin{array}{r|l} & \quad 1 \\ \hline x+2 & x+3 \\ & x+2 \\ \hline & \quad 1 \end{array}$$

$$= \lim_{x \to -2} \left(1 + \frac{1}{x+2}\right)$$ الآن نستطيع رسم الدالة



$\longrightarrow$

$$\lim_{x \to -2} \left(1 + \frac{1}{x+2}\right) = \lim_{x \to -2} \left(\frac{x+3}{x+2}\right)$$

Does not Exist, because

$$\left.\begin{aligned} \lim_{x \to -2^+} \left(\frac{x+3}{x+2}\right) &= +\infty \\ \lim_{x \to -2^-} \left(\frac{x+3}{x+2}\right) &= -\infty \end{aligned}\right\} \text{One-sided Limit not equal}$$

or

$$\lim_{x \to -2^+} \left(\frac{x+3}{x+2}\right) \neq \lim_{x \to -2^-} \left(\frac{x+3}{x-2}\right)$$

Note: See Figure 2.7 (contains on function havent Limit as $x \longrightarrow 0$)

* Example: $\lim_{x \to 0} \frac{\tan(x)}{x}$

solution: $\lim_{x \to 0} \frac{\tan x}{x} = \frac{0}{0}$!!!

$$= \lim_{x \to 0} \left(\frac{\sin x}{x} * \frac{1}{\cos x} \right)$$

$$= 1 * \frac{1}{1}$$

$$= 1$$

Example: $\lim_{x \to 0} \frac{\sin(2x)}{x} = \frac{0}{0}$!!!

$$= 2 \lim_{x \to 0} \frac{\sin(2x)}{2x} = 2(1) = 2$$

Using L'Hopital's Rule

Example: $\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \frac{0}{0}$!!!

$$= \lim_{x \to 0} \frac{\frac{\sin 3x}{x} * \frac{3}{3}}{\frac{\sin 5x}{x} * \frac{5}{5}}$$

$$= \frac{3}{5} \lim_{x \to 0} \frac{\frac{\sin(3x)}{3x}}{\frac{\sin(5x)}{5x}}$$

$$= \frac{3}{5} * \frac{1}{1} = \frac{3}{5}$$

* Limits of trigonometric functions.

* Find $\lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0}$!!!

نبحث عن النهاية بطرق اخرى

$\Rightarrow$ Using L'Hopital 's rule

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

$$\therefore \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos x}{1}$$

$$= \frac{1}{1} = 1$$

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* Example: $\lim_{x \to 0} \frac{1 - \cos x}{x} = \frac{1-1}{0} = \frac{0}{0}$!!!

نبحث عن النهاية بطرق اخرى

$$= \lim_{x \to 0} \left[\frac{1 - \cos x}{x} * \frac{1 + \cos x}{1 + \cos x} \right]$$

$$= \lim_{x \to 0} \left[\frac{1 - \cos^2 x}{x(1 + \cos x)} \right]$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x} * \frac{\sin x}{1 + \cos x} \right)$$

$$= (1) * \frac{0}{2}$$

$$= 0$$

* Example:

Show that $\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$

<u>solution</u>:

Using half angle formula

$$\cos(h) = 1 - 2\sin^2\left(\frac{h}{2}\right)$$

$$\therefore \lim_{h \to 0} \frac{\cancel{1} - 2\sin^2\left(\frac{h}{2}\right) - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{-2\sin^2\left(\frac{h}{2}\right)}{h}$$

$$= \frac{-2}{2} \lim_{h \to 0} \left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} * \sin\left(\frac{h}{2}\right) \right)$$

$$= -1 * (1 * 0)$$

$$= 0 \quad \#$$

Example: $\lim\limits_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$!!!

نبحث عن النهاية بطرق أخرى

Using L'Hopital's rule

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$$

Example: $\lim\limits_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \frac{1}{0} - \frac{1}{0}$!!!

نبحث عن النهاية بطرق أخرى

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0}\left(\frac{\sin x - x}{x \cdot \sin x}\right) = \frac{0}{0} \text{ !!!}$$

$\therefore$ So we need to use L'Hopital's rule:

$$\lim_{x\to 0}\left(\frac{\sin x - x}{x \cdot \sin x}\right) = \lim_{x\to 0}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right) = \frac{0}{0} \text{ !!!}$$

$\therefore$ so we need to use L'Hopital's rule again:

$$\lim_{x\to 0}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right) = \lim_{x\to 0}\left(\frac{-\sin x}{\cos x + (-x\sin x + \cos x)}\right)$$

$$= \frac{0}{2} = 0$$