Derivatives الملازمة الخامسة

The Laws of Derivatives

① If $y = c$ where c is constant

$dy = 0$

② If $y = x$ where x is variable

$dy = dx \Rightarrow \frac{dy}{dx} = y' = 1$

③ If $y = cx$ where c is constant

$dy = c\,dx \Rightarrow y' = c$

ex: $y = 9x \Rightarrow dy = 9\,dx \Rightarrow y' = \frac{dy}{dx} = 9$

④ If $y = x^n$

$\frac{dy}{dx} = n\,x^{n-1}\,dx \Rightarrow y' = n \cdot x^{n-1}$

$\rightarrow$

ex: $y = 10x^5$

$y' = 50x^4$

⑤ If $y = u(x) \pm g(x)$

$y' = u'(x) \pm g'(x)$

ex: $y = x^3 - 6x + 8$

$y' = 3x^2 - 6$

⑥ The derivative of product of two functions.

If u and v are two functions

then :

$y = u \cdot v$

$y' = u \cdot v' + v \cdot u'$

→

ex: $y = x^2 \cdot \sqrt{x^2-2}$, find y'

sol. $y = x^2 \cdot (x^2-2)^{\frac{1}{2}}$

$y' = x^2 \cdot \frac{1}{\cancel{2}}(x^2-2)^{\frac{-1}{2}} \cdot \cancel{2}x + 2x \cdot \sqrt{x^2-2}$

$$y' = \frac{x^3}{\sqrt{x^2-2}} + 2x \cdot \sqrt{x^2-2}$$

⑦ The Quetient Rule:

If $y = \frac{u}{v}$

$$y' = \frac{u' \cdot v - u \cdot v'}{(v)^2}$$

Ex: Derive $f(x) = \frac{x^2-1}{x^4+1}$

solution: $f'(x) = \frac{2x(x^4+1)-(x^2-1) \cdot 4x^3}{(x^4+1)^2}$

Derivative of Logarithmic and Exponential Functions:

(8) If $y = Ln(x)$

$$dy = \frac{1}{x} \cdot dx \Rightarrow y' = \frac{dy}{dx} = \frac{1}{x}$$

(9) If $y = Log_a x$

$$y = \frac{\ln x}{\ln a}$$

$$y' = \frac{1}{x} \cdot \frac{1}{\ln a}$$

$$y' = \frac{1}{x \cdot \ln a}$$

(10) If $y = a^x$ where a is constant positive number.

$$dy = a^x \ln a \cdot dx$$

$$y' = a^x \ln a$$

- If $y = e^x$

$$y' = e^x$$

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* Derivatives of trigonometric functions:

$y = \sin x \Rightarrow dy = \cos x \; dx$

$y = \cos x \Rightarrow dy = -\sin x \; dx$

$y = \tan x \Rightarrow dy = \sec^2 x \; dx$

$y = \cot x \Rightarrow dy = -\csc^2 x \; dx$

$y = \sec x \Rightarrow dy = \sec x \cdot \tan x \; dx$

$y = \csc x \Rightarrow dy = -\csc x \cdot \cot x \; dx$

ex: find $f'(x)$ if $f(x) = \sec(\ln x)$

<u>sol.</u> $f'(x) = \sec(\ln x) \cdot \tan(\ln x) \cdot \frac{1}{x}$

where $f'(x) = \frac{dy}{dx}$

→

ex: find y' if

(a) $y = x^2 . \tan x$

(b) $y = \dfrac{\sin x}{1 + \cos x}$

(c) $y''(\pi/4)$ if

$y(x) = \sec(x)$

solution of (c)

$y(x) = \sec(x)$

$y'(x) = \sec(x) . \tan(x)$

$y''(x) = \sec(x) . \sec^2(x) + \tan(x) . \sec(x) . \tan(x)$

$y''(\frac{\pi}{4}) = \sec^3(\frac{\pi}{4}) + \sec(\frac{\pi}{4}) \tan^2(\frac{\pi}{4})$

$y''(\frac{\pi}{4}) = 2.828 + 1.414$

$y''(\frac{\pi}{4}) = 4.242$

Chain Rule :

If we know the derivatives of f and g, how we can use this information to find the derivative of the composite function $f \circ g$?

Let $y = f(u)$ and $u = g(x)$ then the chain rule is written by:

$$\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$$

Example: Find dy/dx if $y = \tan(u)$

and $u = 4x^3 + x$

<u>solution</u>: $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$

$$\frac{dy}{du} = \sec^2(u)$$

$$\frac{du}{dx} = 12x^2 + 1$$

$$\therefore \frac{dy}{dx} = \sec^2(u) * (12x^2+1)$$

$$\frac{dy}{dx} = \left(\sec^2(4x^3+x)\right) * \left(12x^2+1\right)$$

* Implicit differentiation : الدوال الضمنية

هي الدوال التي لا يمكن فيها التعبير عن أحد المتغيرات بدلالة الآخر

for example:

$$x^3 + yx^2 + y^3 = 8$$

Example: Use the implicit differentiation to find dy/dx if $5y^2 + \sin y = x^2$

solution:

$$\frac{d}{dx}(5y^2 + \sin y) = \frac{d}{dx}(x^2)$$

$$10y \frac{dy}{dx} + \cos y \cdot \frac{dy}{dx} = 2x \cdot \cancel{\frac{dx}{dx}}$$

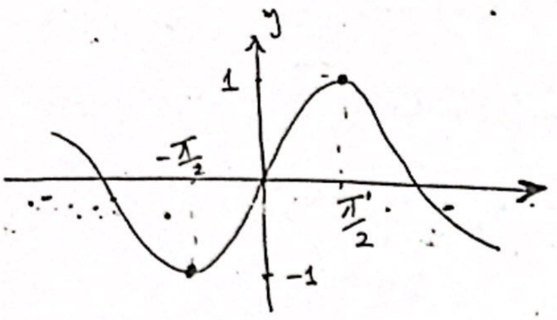
$$10y \cdot y' + y' \cdot \cos y = 2x$$

$$y'(10y + \cos y) = 2x$$

$$y' = \frac{2x}{10y + \cos y}$$

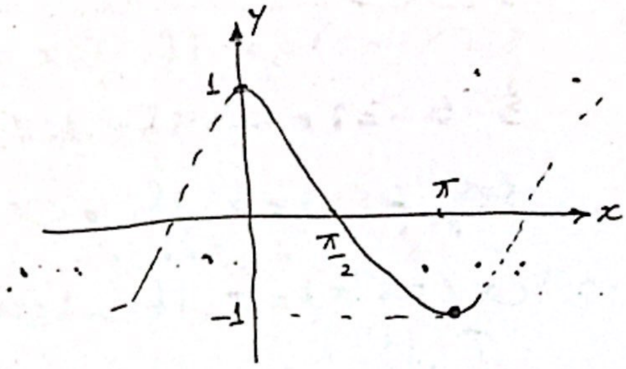
Example: Find d^2y/dx^2 if $4x^2 - 2y^2 = 9$

Inverse of Trigonometric functions:-



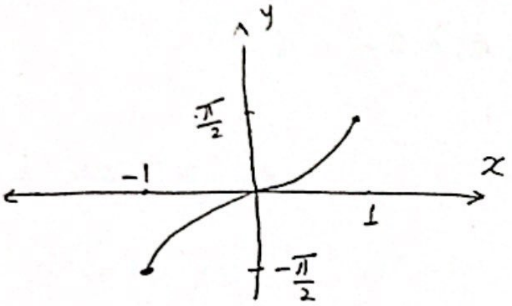
$y = \sin x$

$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$



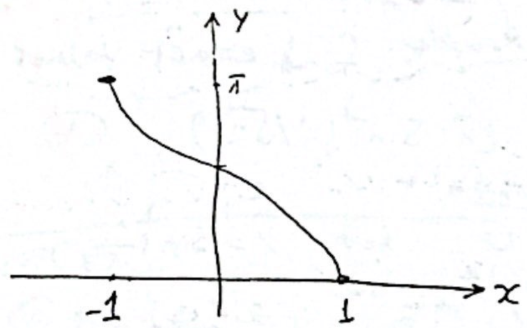
$y = \cos x$

$0 \le x \le \pi$



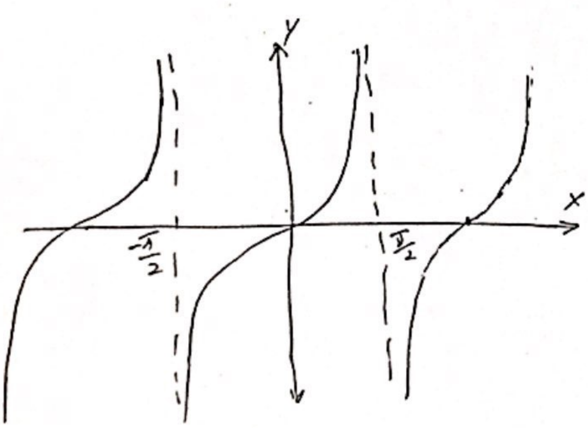
$y = \sin^{-1} x$

$-1 \le x \le 1$

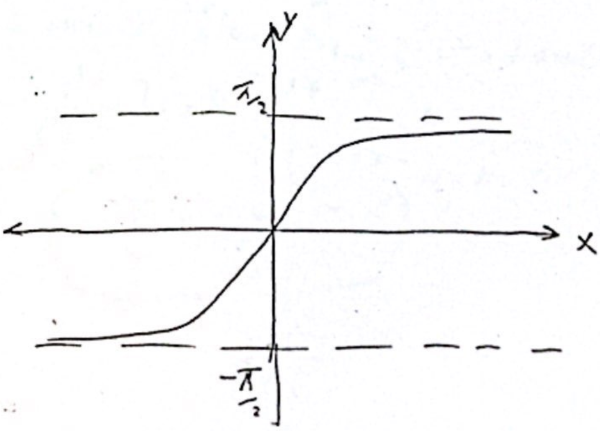


$y = \cos^{-1} x$

$-1 \le x \le 1$



$\Rightarrow$



$y = \tan x$

$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$y = \tan^{-1} x$

$-\infty < x < +\infty$

Basic Relationships :-

$\sin^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \leq x \leq \pi/2$

$\sin(\sin^{-1} x) = x$ if $-1 \leq x \leq 1$

$\cos^{-1}(\cos x) = x$ if $0 \leq x \leq \pi$

$\cos(\cos^{-1} x) = x$ if $-1 \leq x \leq 1$

$\tan^{-1}(\tan x) = x$ if $-\frac{\pi}{2} < x < \pi/2$

$\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$

Example: find exact values of

ⓐ $\sin^{-1}(1/\sqrt{2})$ ⓑ $\sin^{-1}(-1)$

Solution:

ⓐ let $y = \sin^{-1}(\frac{1}{\sqrt{2}})$

$\sin y = \sin(\sin^{-1}(\frac{1}{\sqrt{2}}))$

$\sin y = \frac{1}{\sqrt{2}}$

from triangle

$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore \sin^{-1} \sin(\frac{\pi}{4}) = \sin^{-1} \frac{1}{\sqrt{2}}$

$\therefore \sin^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$

Ex: Show that $\sin^{-1}(-x) = -\sin^{-1}(x)$

Sol: Let $y = \sin^{-1}(-x)$

$$\sin y = \sin \sin^{-1}(-x)$$

$$\sin y = -x$$

$$\therefore x = -\sin y$$

$$x = \sin(-y)$$

$$\sin^{-1}(x) = \sin^{-1}\sin(-y)$$

$$\sin^{-1}(x) = -y$$

$$\therefore y = -\sin^{-1}(x)$$

Exercises:-

① Show that

ⓐ $\sec^{-1}(x) = \cos^{-1}(1/x)$ if $|x| \geq 1$

ⓑ $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

ⓒ $\csc^{-1}x = \sin^{-1}(\frac{1}{x})$ if $|x| \geq 1$

② Given that $\theta = \cos^{-1}(\frac{1}{2})$, find the exact values of $\sin\theta$, $\tan\theta$, $\sec\theta$, and $\csc\theta$.

③ Given that $\theta = \tan^{-1}(\frac{4}{3})$, find the exact values of $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

④ Find the exact value of

ⓐ $\sin^{-1}(\frac{\sqrt{3}}{2})$. ⓑ $\cos^{-1}(\frac{1}{2})$

ⓒ $\sec^{-1}(-2)$ ⓒ $\tan^{-1}(-1)$

⑤ find

ⓐ $\sin(\sin^{-1}(\frac{2}{3}) + \cos^{-1}(\frac{1}{3}))$

ⓑ $\sin(\cos^{-1}(\frac{1}{5}) - \tan^{-1}(2))$

ⓒ $\cos(2\sin^{-1}(\frac{5}{13}))$

ⓓ $\tan(2\sec^{-1}(\frac{5}{4}))$

Derivatives of The inverse Trigonometric functions:

① $y = \sin^{-1} x \Rightarrow dy = \dfrac{1}{\sqrt{1-x^2}}\, dx$

② $y = \cos^{-1} x \Rightarrow dy = \dfrac{-1}{\sqrt{1-x^2}}\, dx$

③ $y = \tan^{-1} x \Rightarrow dy = \dfrac{1}{1+x^2}\, dx$

④ $y = \cot^{-1} x \Rightarrow dy = \dfrac{-1}{1+x^2}\, dx$

⑤ $y = \sec^{-1} x \Rightarrow dy = \dfrac{1}{|x|\sqrt{x^2-1}}\, dx \qquad (1 < |x|)$

⑥ $y = \csc^{-1} x \Rightarrow dy = \dfrac{-1}{|x|\sqrt{x^2-1}}\, dx$

Ex: show that $\dfrac{d}{dx}[\sin^{-1} x] = \dfrac{1}{\sqrt{1-x^2}}$

Solution.

Let $y = \sin^{-1} x$

$\Rightarrow \sin y = \sin(\sin^{-1} x)$

$\Rightarrow \sin y = x$

$\Rightarrow \cos y\, dy = dx$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos y}$

$\therefore y' = \dfrac{1}{\sqrt{1-x^2}}$