Lecture 6: The Derivative as a Rate of Change

مدرس المادة: م.م. أحمد مؤيد عبدالحسين

جامعة الفرات الأوسط التقنية / الكلية التقنية الهندسية / نجف

Review

DEFINITION Average Rate of Change over an Interval

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

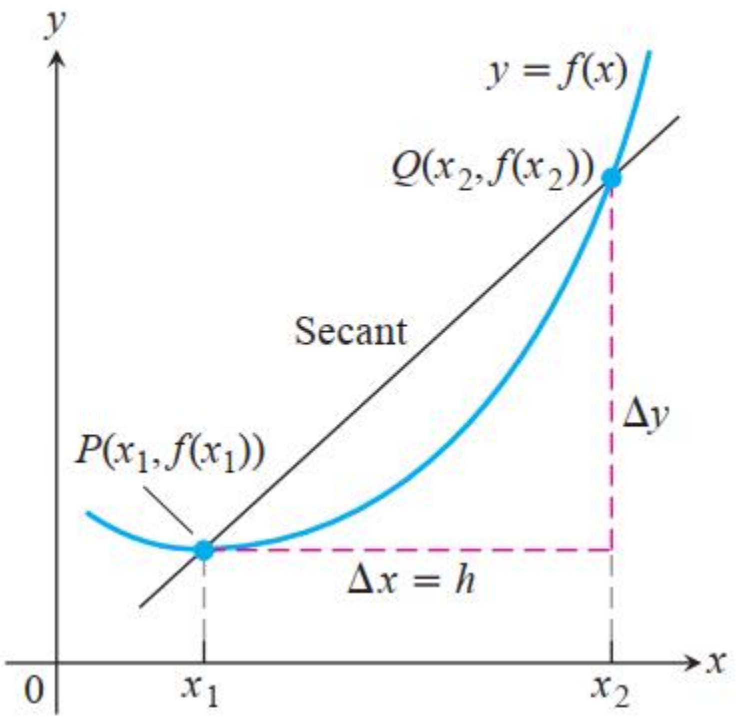


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

EXAMPLE 3 The Average Growth Rate of a Laboratory Population

Figure 2.2 shows how a population of fruit flies (*Drosophila*) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time, and the points joined by a smooth curve (colored blue in Figure 2.2). Find the average growth rate from day 23 to day 45.

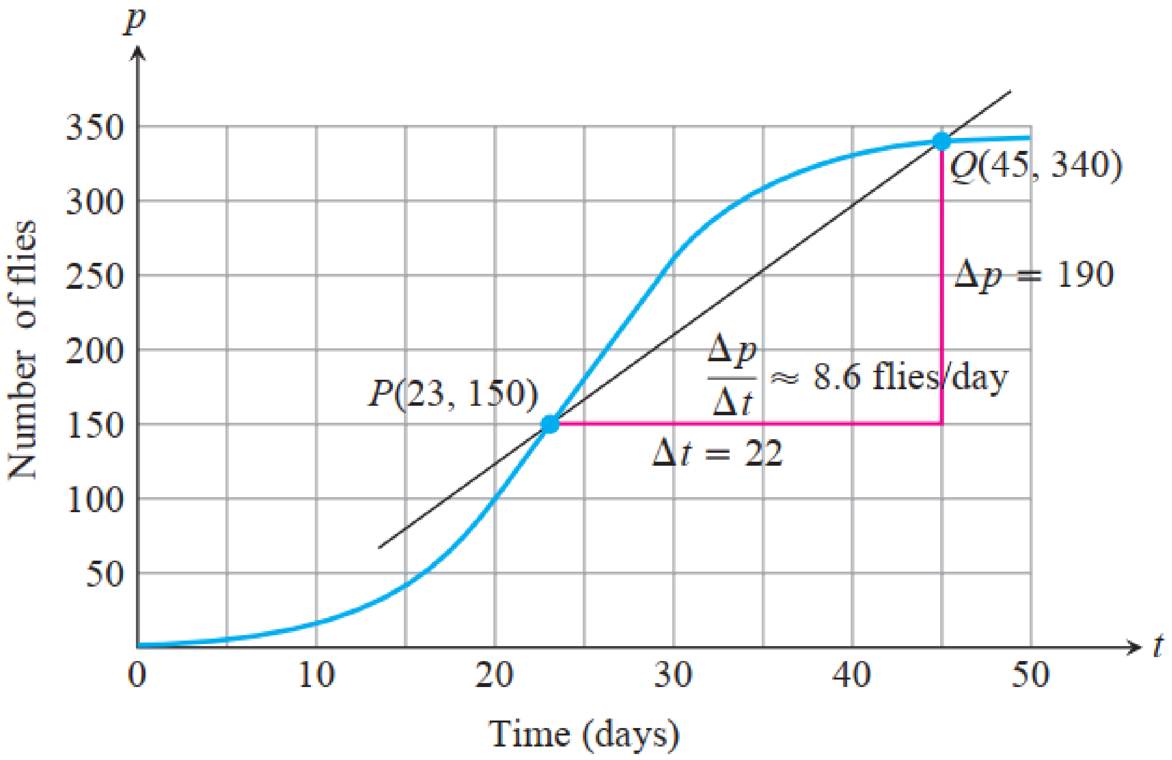


FIGURE 2.2 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line.

The Derivative as a Rate of Change

Instantaneous Rates of Change

If we interpret the difference quotient $(f(x + h) - f(x))/h$ as the average rate of change in f over the interval from x to $x + h$, we can interpret its limit as $h \to 0$ as the rate at which f is changing at the point x.

DEFINITION Instantaneous Rate of Change

The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

The Derivative as a Rate of Change

EXAMPLE 1 How a Circle's Area Changes with Its Diameter

The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4} D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?

The Derivative as a Rate of Change

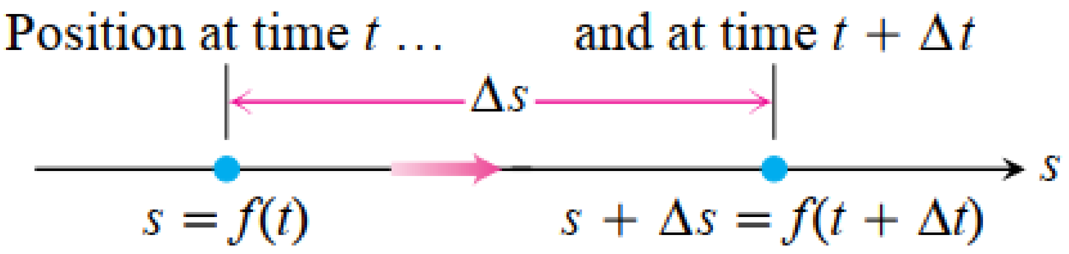


FIGURE 3.12 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$.

and the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

DEFINITION Velocity

Velocity (instantaneous velocity) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

EXAMPLE 2 Finding the Velocity of a Race Car

Figure 3.13 shows the time-to-distance graph of a 1996 Riley & Scott Mk III-Olds WSC race car. The slope of the secant PQ is the average velocity for the 3-sec interval from $t = 2$ to $t = 5$ sec; in this case, it is about 100 ft/sec or 68 mph.

The slope of the tangent at P is the speedometer reading at $t = 2$ sec, about 57 ft/sec or 39 mph. The acceleration for the period shown is a nearly constant 28.5 ft/sec^2 during

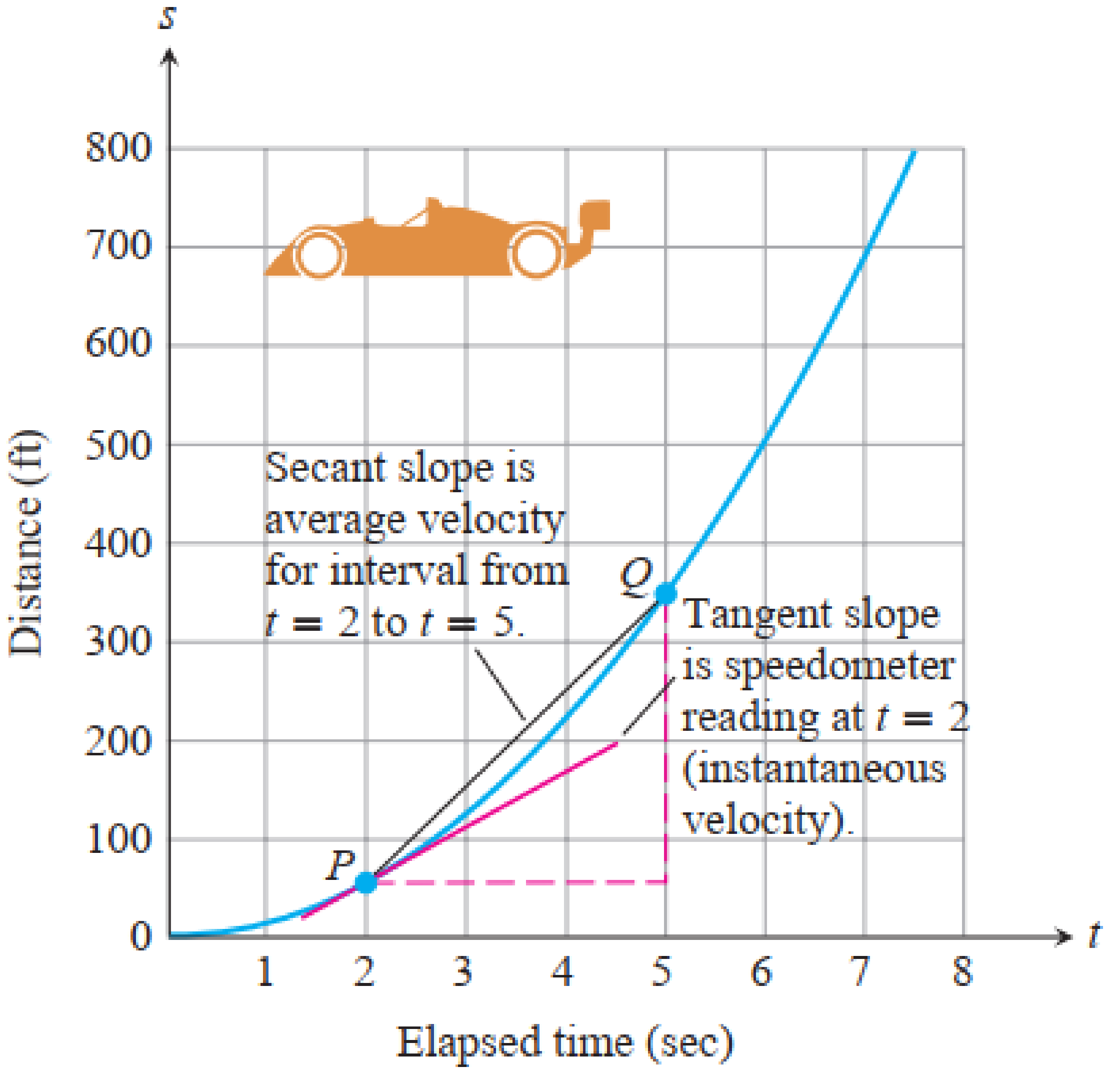


FIGURE 3.13 The time-to-distance graph for Example 2. The slope of the tangent line at P is the instantaneous velocity at $t = 2$ sec.

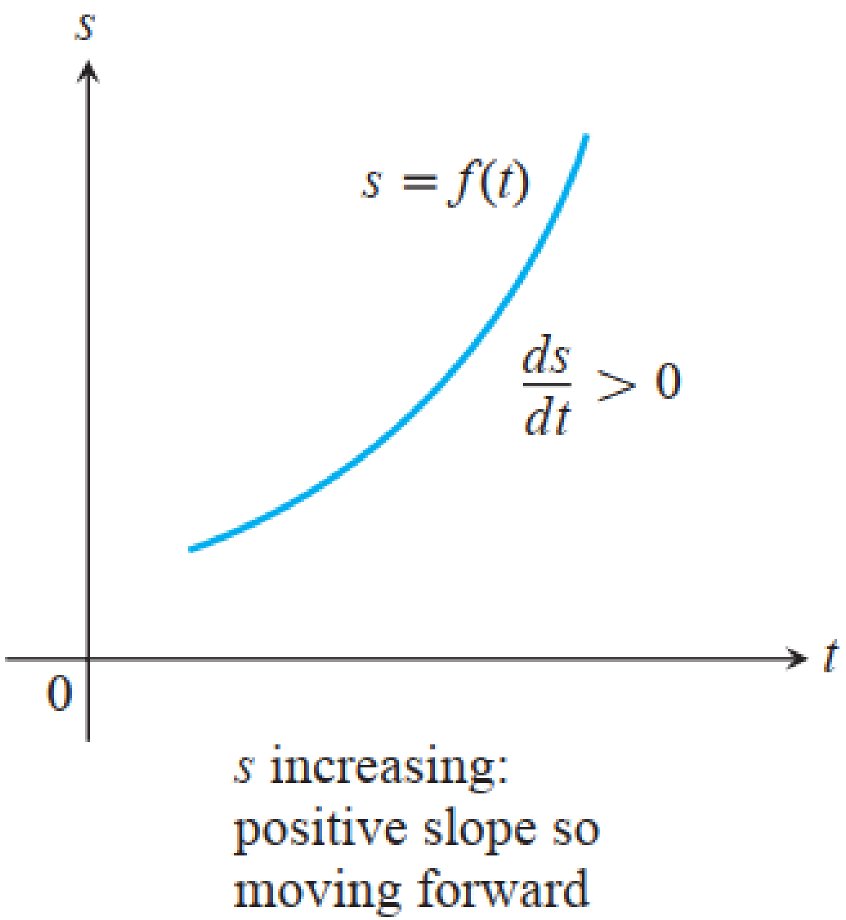


FIGURE 3.14 For motion $s = f(t)$ along a straight line, $v = ds/dt$ is positive when s increases and negative when s decreases.

Example 3

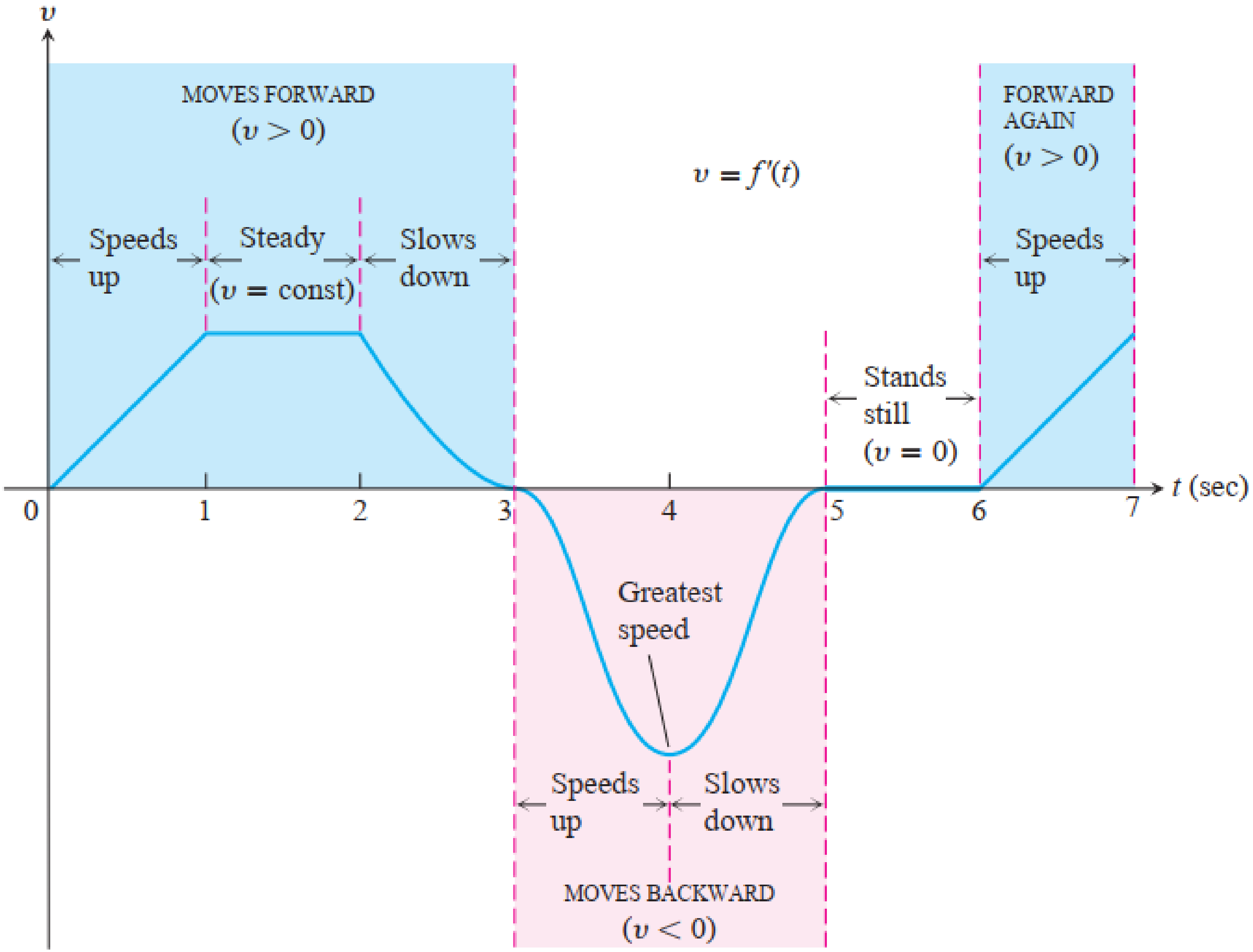


FIGURE 3.15 The velocity graph for Example 3.

DEFINITIONS Acceleration,

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

EXAMPLE 4 Modeling Free Fall

Figure 3.16 shows the free fall of a heavy ball bearing released from rest at time $t = 0$ sec.

(a) How many meters does the ball fall in the first 2 sec?

(b) What is its velocity, speed, and acceleration then?

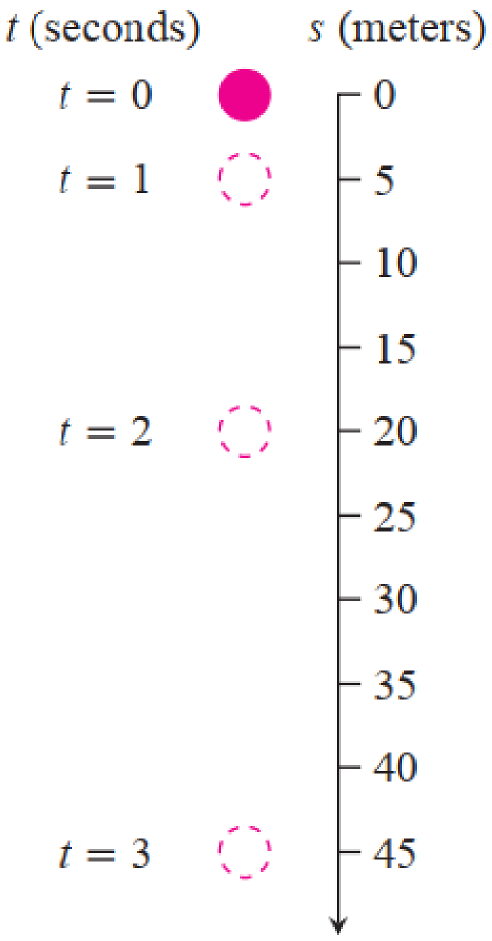


FIGURE 3.16 A ball bearing falling from rest (Example 4).

EXAMPLE 5 Modeling Vertical Motion

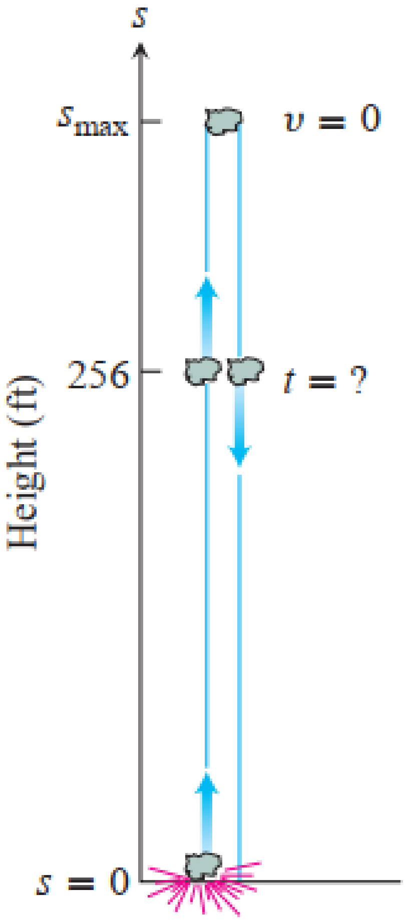
A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Figure 3.17a). It reaches a height of $s = 160t - 16t^2$ ft after t sec.

(a) How high does the rock go?

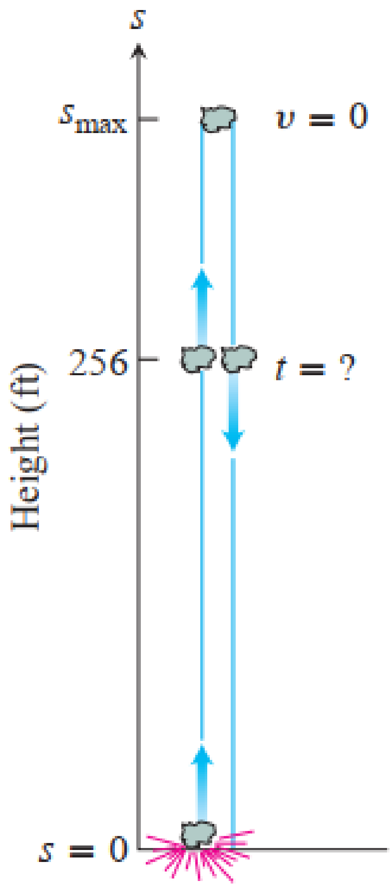
(b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground again?



(a)



(a)

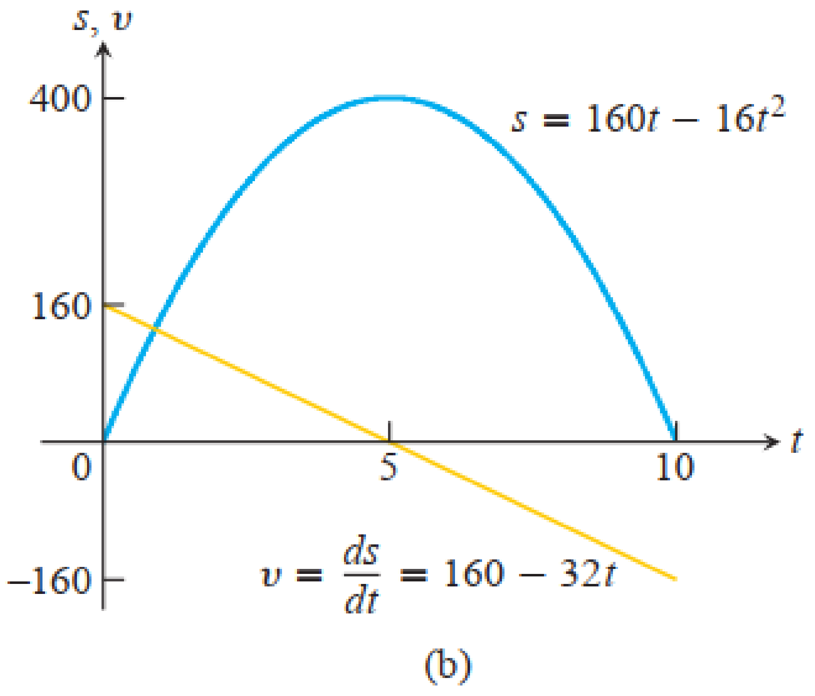


FIGURE 3.17 (a) The rock in Example 5. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

EXAMPLE 15 Dropping Emergency Supplies

A Red Cross aircraft is dropping emergency food and medical supplies into a disaster area. If the aircraft releases the supplies immediately above the edge of an open field 700 ft long and if the cargo moves along the path

$$x = 120t \qquad \text{and} \qquad y = -16t^2 + 500, \qquad t \geq 0$$

does the cargo land in the field? The coordinates x and y are measured in feet, and the parameter t (time since release) in seconds. Find a Cartesian equation for the path of the falling cargo (Figure 3.32) and the cargo's rate of descent relative to its forward motion when it hits the ground.

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3}\pi r^3.$$

Using the Chain Rule, we differentiate to find the related rates equation

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

EXAMPLE 2 A Rising Balloon

A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

End of this lecture

Any Questions ?