المحاضرة الثامنة: التكاملات =

*Indefinite Integrals

→ Some basic rules :

① $\int du = u + c$ where c is constant

② $\int u^n \, du = \dfrac{u^{n+1}}{n+1} + C \quad , n \neq -1$

③ $\int \dfrac{du}{u} = \ln|u| + C$

④ $\int a^u \cdot du = \dfrac{a^u}{\ln a} + C$

⑤ $\int e^u \cdot du = e^u + C$

*·Integration of trigonometric Functions:

① $\int \sin(u)\,du = -\cos(u) + c$

② $\int \cos(u)\,du = \sin(u) + c$

③ $\int \tan(u)\,du = \int \frac{\sin u}{\cos u}\,du$

$= -\ln|\cos u| + C = \ln|\sec u| + c$

④ $\int \cot(u)\,du = \int \frac{\cos u}{\sin u} \cdot du = \ln|\sin u| + C$

⑤ $\int \sec^2(u)\,du = \tan(u) + c$

⑥ $\int \csc^2(u)\,du = -\cot(u) + c$

⑦ $\int \sec(u) \cdot \tan(u)\,du = \sec(u) + c$

⑧ $\int \csc(u) \cdot \cot(u) \cdot du = -\csc(u) + c$

* Integration methods

① Directly :

Example ①: $\int \sin(7x)\,dx$

$$= -\frac{\cos(7x)}{7} + C$$

Example ②: $\int x^4\,dx = \frac{x^5}{5} + C$

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② By substitution :

Example ①: $\int \sqrt{1+y^2}\,.\,(2y)\,dy$

Let $u = 1 + y^2$

$$du = 2y\,dy$$

$$2y = \frac{du}{dy}$$

$$\therefore \int \sqrt{u}\,.\,\frac{du}{dy}\,.\,dy$$

$$= \int (u)^{\frac{1}{2}}\,du$$

$$= \frac{u^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{2}{3}(1+y^2)^{3/2} + C$$

⇒ Example ②: $\int \frac{3}{3} x^2 \sin(x^3)\, dx$

Let: $u = x^3$

$du = 3x^2\, dx$

$3x^2 = \frac{du}{dx}$

$= \frac{1}{3} \int \frac{du}{dx} \cdot \sin(u) \cdot dx$

$= \frac{1}{3} \int \sin(u) \cdot du$

$= \frac{1}{3} \left(-\cos(u)\right) + C$

$= \frac{-\cos(x^3)}{3} + C$

* Note: You can solve more integrals by use substitution method @ Calculas Section 5.5, P:368

(3) Integration by Parts:

Integration by Parts is a method for simplifying integrals of the form:

$$\int f(x) \cdot g(x)\, dx$$

Formula:

- $$\int f(x) \cdot g'(x)\, dx = f(x)\, g(x) - \int f'(x) \cdot g(x)\, dx$$

$$\int u\, dv = u \cdot v - \int v\, du$$

→

*Example ①: $\int x\cos(x)\,dx$

Let $u = x$ $\qquad$ $dv = \cos(x)dx$

$du = dx$ $\qquad$ $V = \sin(x)$

$-\int$

$$\therefore \int x\cos(x)\,dx = x\cdot\sin(x) - \int \sin(x)\,dx$$

$$= x\cdot\sin(x) + \cos(x) + c$$

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④. Tabular Integration

* Example ①:

طريقة الحل الأولى
by parts

$$\int x^2 e^x\,dx$$

Let: $u = x^2$ $\qquad$ $dv = e^x\cdot dx$

$du = 2x\,dx$ $\qquad$ $V = e^x$

$-\int$

$$\int x^2\cdot e^x\,dx = x^2\cdot e^x - 2\int x e^x\cdot dx$$

Let: $u = x$ $\qquad$ $dV = e^x\cdot dx$

$du = dx$ $\qquad$ $V = e^x$

$-\int$

$\Longrightarrow$

$$\Rightarrow \int x^2 \cdot e^x \, dx = x^2 \cdot e^x - 2\left(xe^x - \int e^x \cdot dx\right)$$

$$= x^2 \cdot e^x - 2x \cdot e^x + 2e^x + C$$

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by tabular integration طريقة الحل الثانية

$f(x)$		$g(x)$
x^2		e^x
	+	e^x
$2x$	−	e^x
2	+	e^x
0		

$$\int x^2 \cdot e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C$$

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* Example ②: $\int x^3 \cdot \sin x \, dx$

$f(x)$		$g(x)$
		$\sin(x)$
x^3	+	$-\cos(x)$
$3x^2$	−	$-\sin(x)$
$6x$	+	$\cos(x)$
6	−	$+\sin(x)$
0		

$$\int x^3 \cdot \sin(x) \, dx = -x^3 \cdot \cos(x) + 3x^2 \sin(x) + 6x\cos(x) - 6\sin(x) + C$$

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*Note: You can solve more integrals by use Integration by parts method @ Calculas Seetion 8.2, P: 561

⑤ Integral of Rational functions

a.) Partial fraction

* نستخدم هذه الطريقة عندما يكون درجة البسط أصغر من المقام

Example ①: $\int \frac{1}{x(x-1)^2}\,dx$

*(دائما درجة البسط أقل بدرجة من المقام الذي داخل القوس)

Let:
$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)}$$

$(x-1)^2 \qquad x \qquad x(x-1)$

$$1 = A(x-1)^2 + Bx + Cx(x-1)$$

$$1 = Ax^2 - 2Ax + A + Bx + Cx^2 - Cx$$

$$1 = x^2(A+C) + x(-2A+B-C) + A$$

$A + C = 0 \quad \ldots\ldots ①$

$-2A + B - C = 0 \quad \ldots\ldots ②$

$\boxed{A = 1} \quad \ldots\ldots ③$

$\therefore \boxed{C = -1}$

$\boxed{B = 1}$ من خلال التعويض بالمعادلة ②

$\longrightarrow$

$$\therefore \int \frac{dx}{x(x-1)^2} = \int \frac{1}{x}dx + \int \frac{1}{(x-1)^2}dx - \int \frac{dx}{x-1}$$

$$= \ln|x| + \frac{-1}{(x-1)^{+1}} - \ln|x-1| + c$$

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* Example ②: $\int \frac{5x-3}{x^2-2x-3}\,dx$

ملاحظة: المزيد من الأمثلة والأسئلة حول التكامل بتجزئة الكسر
(Partial Fraction) موجودة في section 8.3 ص 570

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* Example ③: $\int \frac{dx}{x(x^2-1)}$

Let: $\frac{1}{x(x^2-1)} = \frac{A}{x} + \frac{Bx+C}{(x^2-1)}$

(x^2-1) $\quad$ x

$$1 = Ax^2 - A + Bx^2 + Cx$$

$\therefore A + B = 0$

$\boxed{C = 0}$

$\boxed{-A = 1}$

$A = -1$

$\therefore \boxed{B = +1}$

$\Longrightarrow$

$$\Rightarrow \therefore \int \frac{dx}{x(x^2-1)} = \int \frac{-dx}{x} + \frac{1}{2}\int \frac{2x}{x^2-1}dx$$

$$= -\ln|x| + \frac{1}{2}\ln|x^2-1| + C$$

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*Example (4): $\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}dx$

$$\int \frac{x^3+x^2+x+2}{(x^2+1)(x^2+2)}dx$$

Let : $\frac{x^3+x^2+x+2}{(x^2+1)(x^2+2)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+2)}$

x^2+2 x^2+1

$$x^3+x^2+x+2 = \underline{A}x^3+2Ax+Bx^2+2B+\underline{C}x^3+Cx$$
$$+Dx^2+D$$

$\therefore$ $A+C=1$ .... (1)

$B+D=1$ .... (2)

$2A+C=1$ .... (3)

$2B+D=2$ .... (4)

solve (3) − (1)

$\boxed{A=0} \Rightarrow \boxed{C=1}$

solve (4) − (2)

$\boxed{B=1} \Rightarrow \boxed{D=0}$

باقي الحل

H.W.

⑤ b.) Long Division

$$\int \frac{\text{Polynomial}}{\text{Polynomial}}$$

degree numerator $\geq$ denominator

Example ①: $\displaystyle\int \frac{u^3-1}{u-1}\,du$

بما انه درجة البسط أكبر من درجة المقام فمن الممكن اجراء قسمة طويلة لهذا الكسر.

الناتج (u^2+u+1)

$$\begin{array}{r|l}
 & u^2+u+1 \\
\hline
u-1 & u^3-1 \\
 & \ominus u^3 \oplus u^2 \\
\hline
 & u^2-1 \\
 & u^2-u \\
\hline
 & u-1 \\
 & u-1 \\
\hline
 & 0
\end{array}$$

لا يوجد باقي $\leftarrow 0$

$$\therefore \int \frac{u^3-1}{u-1}\,du = \int (u^2+u+1)\,du = \frac{u^3}{3}+\frac{u^2}{2}+u+c$$

* Example ②: $\int \frac{x^2+2}{x+2} dx$

ناتج القسمة $(x-2)$

$$
\begin{array}{r|l}
x+2 & x^2+2 \\
 & x^2+2x \\ \hline
 & -2x+2 \\
 & -2x-4 \\ \hline
\end{array}
$$

الباقي $\Rightarrow 6$

$$\therefore \int \frac{x^2+2}{x+2} dx = \int \left(x-2+\frac{6}{x+2}\right) dx$$

$$= \frac{x^2}{2} - 2x + 6 \ln|x+2| + C$$

⑤ c.) Other Integral techniques

$$\int \frac{1}{a^2 + x^2}\, dx \qquad \text{Let } x = a \cdot \tan\theta$$

$$\int \frac{1}{a^2 - x^2}\, dx \qquad \text{Let } x = a \cdot \sin\theta$$

$$\int \frac{1}{x^2 - a^2}\, dx \qquad \text{Let } x = a \cdot \sec\theta$$

* Example ①: $\int \frac{dx}{\sqrt{25-16x^2}}$

$$= \frac{1}{4}\int \frac{dx}{\sqrt{\frac{25}{16} - x^2}}$$

Let : $x = \frac{5}{4}\sin\theta \implies \theta = \sin^{-1}\left(\frac{4}{5}x\right)$

$$dx = \frac{5}{4}\cos\theta \, d\theta$$

$$= \frac{1}{4}\int \frac{\frac{5}{4}\cos\theta \, d\theta}{\sqrt{\frac{25}{16} - \frac{25}{16}\sin^2\theta}}$$

$$= \frac{1}{4}\int \frac{\cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \frac{1}{4}\int d\theta$$

$$= \frac{1}{4}\theta + C$$

$$= \frac{1}{4}\sin^{-1}\left(\frac{4}{5}x\right) + C$$