REPUBLIC OF IRAQ
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
AL-FURAT AL-AWSAT TECHNICAL UNIVERSITY
ENGINEERING TECHNICAL COLLEGE- NAJAF
AVIONIC TECHNIQUES ENGINEERING DEPARTMENT

ANALOG AND DIGITAL COMMUNICATION SYSTEMS
For 3rd Class Students

Assistant lecturer

Amjed Adnan Al-mudaffer

Fourier Series

Theory

A graph of periodic function $f(x)$ that has period L exhibits the same pattern every L units along the x-axis, so that $f(x + L) = f(x)$ for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods).

The Fourier Series of periodic function $f(t)$ is a representation that resolve $f(t)$ into average component (DC) and alternative component (AC) comprising infinite series and harmonic

Periodic Functions

If the value of each ordinate $f(t)$ repeats itself at equal intervals in the abscissa, then $f(t)$ is said to be a periodic function

If $f(t) = f(t + T) = f(t + 2T) = \ldots. f(t + nT)$ then ***T*** is called the period of the function f (t).

For example : .

$\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = ..$. so $\sin x$ is a periodic function with the period 2π. This is also called sinusoidal periodic function.

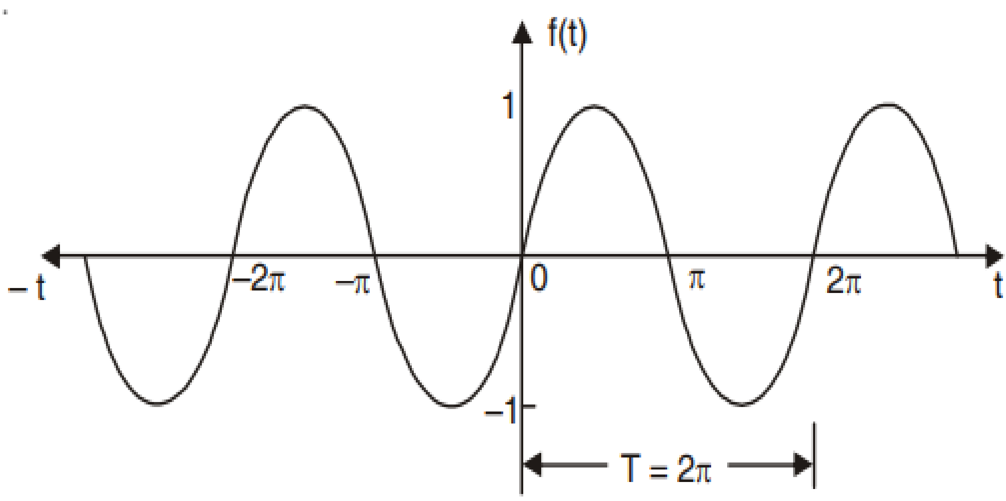


Fig.1 sinusoidal periodic function

Fourier Series

Here we will express a non-sinusoidal periodic function into a fundamental and its harmonics. A series of sines and cosines of an angle and its multiples of the form.

$$f(t) = \frac{a_0}{2} + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots + b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \cdots$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

- The $\cos(nx)$ or $\sin(nx)$ is called n^{th} harmonic of $f(x)$.
- a_0 is average (DC) component, average value $f(x)$.
- a_n and b_n are the Fourier coefficient.
- $n = 1,2,3,4,5,6, \ldots ..$

USEFUL INTEGRALS

The following integrals are useful in Fourier Series.

(i) $\int_0^{2\pi} \sin nx\, dx = 0$

(ii) $\int_0^{2\pi} \cos nx\, dx = 0$

(iii) $\int_0^{2\pi} \sin^2 nx\, dx = \pi$

(iv) $\int_0^{2\pi} \cos^2 nx\, dx = \pi$

(v) $\int_0^{2\pi} \sin nx \cdot \sin mx\, dx = 0$

(vi) $\int_0^{2\pi} \cos nx \cos mx\, dx = 0$

(vii) $\int_0^{2\pi} \sin nx \cdot \cos mx\, dx = 0$

(viii) $\int_0^{2\pi} \sin nx \cdot \cos nx\, dx = 0$

(ix) $\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$

where $v_1 = \int v\, dx,\ v_2 = \int v_1\, dx$ and so on $u' = \frac{du}{dx},\ u'' = \frac{d^2u}{dx^2}$ and so on and

(x) $\sin n\pi = 0,\ \cos n\pi = (-1)^n$ where $n \in I$

The following are some properties of these functions :

1) $Sin^2\theta + Cos^2\theta = 1$

2) $1 + tan^2\theta = sec^2\theta$ and $1 + Cot^2\theta = csc^2\theta$

3) $Sin(\theta \mp \beta) = Sin\theta.Cos\beta \mp Cos\theta.Sin\beta$

4) $Cos(\theta \mp \beta) = Cos\theta.Cos\beta \pm Sin\theta.Sin\beta$

5) $tan(\theta \mp \beta) = \frac{tan\theta \mp tan\beta}{1 \pm tan\theta.tan\beta}$

6) $Sin2\theta = 2Sin\theta.Cos\theta$ and $Cos2\theta = Cos^2\theta - Sin^2\theta$

7) $Cos^2\theta = \frac{1 + Cos2\theta}{2}$ and $Sin^2\theta = \frac{1 - Cos2\theta}{2}$

8) $Sin(\theta \mp \frac{\pi}{2}) = \mp Cos\theta$ and $Cos(\theta \mp \frac{\pi}{2}) = \pm Sin\theta$

9) $Sin(-\theta) = -Sin\theta$ and $Cos(-\theta) = Cos\theta$ and $tan(-\theta) = -tan\theta$

10) $Sin\theta.Sin\beta = \frac{1}{2}[Cos(\theta - \beta) - Cos(\theta + \beta)]$

$Cos\theta.Cos\beta = \frac{1}{2}[Cos(\theta - \beta) + Cos(\theta + \beta)]$

$Sin\theta.Cos\beta = \frac{1}{2}[Sin(\theta - \beta) + Sin(\theta + \beta)]$

$\cos(2n\pi) = 1$	$\sin(2n\pi) = 0$
$\sin(n\pi) = 0$	$\cos(n\pi) = (-1)^n$
$\sin\left(\frac{n\pi}{2}\right) = \begin{cases}(-1)^{\frac{n-1}{2}} & ,n = odd \\ 0 & ,n = even\end{cases}$	$\cos\left(\frac{n\pi}{2}\right) = \begin{cases}(-1)^{\frac{n}{2}} & ,n = even \\ 0 & ,n = odd\end{cases}$
$e^{\pm jx} = \cos x \pm j \sin x$	$e^{j2n\pi} = 1$

We need to work out the Fourier coefficients (a_0, a_n and b_n) for given functions $f(x)$. This process is broken down into three steps

STEP ONE

Find a_0 integrate both sides from $x = 0$ **to** $x = 2\pi$

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} cos(nx)dx + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} sin(nx)dx$$

$$\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}[2\pi - 0]$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

STEP TWO

Find a_n Multiply each side by $cos(nx)$ and integrate both sides from $x = 0$ **to** $x = 2\pi$

$$\int_0^{2\pi} f(x)\,cos(nx)\,dx = \int_0^{2\pi} \frac{a_0}{2} cos(nx)\,dx + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} cos(nx)\,cos(nx)\,dx + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} sin(nx)\,cos(nx)\,dx$$

$$\int_0^{2\pi} f(x)\,cos(nx)\,dx = 0 + a_n\pi + 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,cos(nx)\,dx$$

STEP THREE

Find b_n Multiply each side by $sin(nx)$ and integrate both sides from $x = 0$ **to** $x = 2\pi$

$$\int_0^{2\pi} f(x)\sin(nx)\,dx = \int_0^{2\pi} \frac{a_0}{2}\sin(nx)\,dx + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} cos(nx)\sin(nx)\,dx + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} sin(nx)\sin(nx)\,dx$$

$$\int_0^{2\pi} f(x)\sin(nx)\ dx = 0 + 0 + b_n\pi$$

$$\mathbf{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\ dx}$$

Example . Find the Fourier series representing

$$f(x) = x \qquad 0 < x < 2\pi$$

and sketch its graph from $x = -4\pi\ to\ x = 4\pi$

Solution

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

Hence $\quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

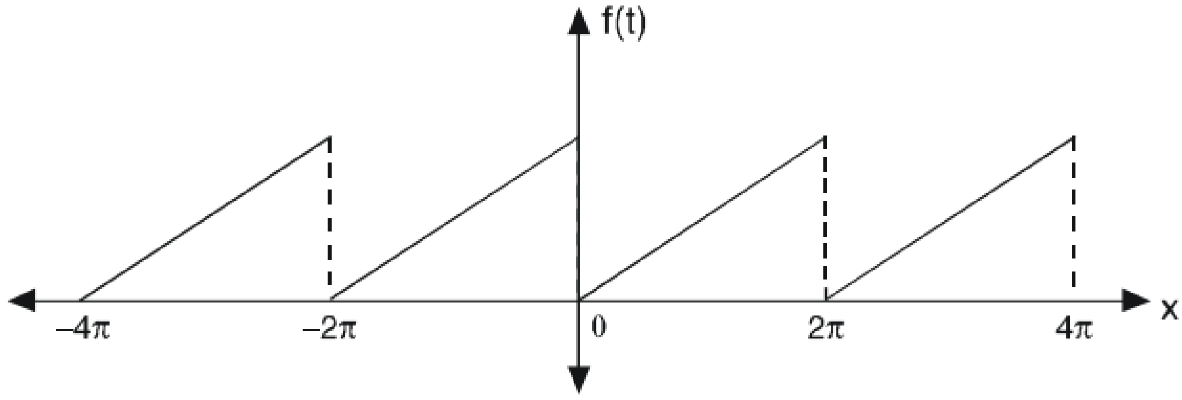
$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

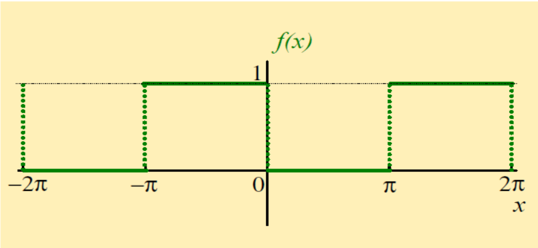
$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi\cos 2n\pi}{n}\right] = -\frac{2}{n}$$

Substituting the values of a_0, a_n, b_n in (1), we get

$$x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right]$$ **Ans.**



Example . Find the Fourier series representing



$$\begin{aligned} a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot \mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot \mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} \mathrm{d}x \\ &= \frac{1}{\pi}[x]_{-\pi}^{0} \\ &= \frac{1}{\pi}(0-(-\pi)) \\ &= \frac{1}{\pi}\cdot(\pi) \\ \text{i.e. } a_0 &= 1\ . \end{aligned}$$

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\cos nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} = \frac{1}{n\pi}[\sin nx]_{-\pi}^{0} \\ &= \frac{1}{n\pi}(\sin 0 - \sin(-n\pi)) \\ &= \frac{1}{n\pi}(0+\sin n\pi) \\ \text{i.e. } a_n &= \frac{1}{n\pi}(0+0) = 0. \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\sin nx\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned} \text{i.e. } b_n &= \frac{1}{\pi}\int_{-\pi}^{0}\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} \\ &= -\frac{1}{n\pi}[\cos nx]_{-\pi}^{0} = -\frac{1}{n\pi}(\cos 0 - \cos(-n\pi)) \\ &= -\frac{1}{n\pi}(1-\cos n\pi) = -\frac{1}{n\pi}(1-(-1)^n),\ \text{see TRIG} \\ \text{i.e. } b_n &= \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n\pi} & , n \text{ odd} \end{cases} \quad, \text{since } (-1)^n = \begin{cases} 1 & , n \text{ even} \\ -1 & , n \text{ odd} \end{cases} \end{aligned}$$

Substituting our results now gives the required series

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]$$

Example . . *Find the Fourier series of the function*

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ +1 & \text{for} \quad \frac{\pi}{2} < x < \pi \end{cases}$$

Solution. Let $\quad f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots \qquad \ldots(1)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 0dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 1\,dx$$

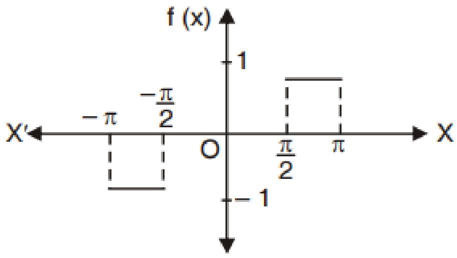
$$= \frac{1}{\pi}\left[-x\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[x\right]_{\pi/2}^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2} - \pi - \frac{\pi}{2}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\ dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\cos nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\cos nx\ dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\cos nx\,dx$$

$$= -\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi} = -\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n} + \frac{\sin n\pi}{n}\right] + \frac{1}{\pi}\left[\frac{\sin n\pi}{n} - \frac{\sin\frac{n\pi}{2}}{n}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$



$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\sin nx\,dx$$

$$+ \frac{1}{n}\int_{\pi/2}^{\pi}(1)\sin nx\,dx$$

$$= \pi\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} - \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] - \frac{1}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$b_1 = \frac{2}{\pi}, \qquad b_2 = -\frac{2}{\pi}, \qquad b_3 = \frac{2}{3\pi}$$

Putting the values of a_0, a_n, b_n in (1) we get $f(x) = \frac{1}{\pi}\left[2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \ldots\right]$ **Ans.**

Example 4. *Find the Fourier series for the periodic function*

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

$$f(x + 2\pi) = f(x)$$

Solution. Let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_0 \cos 2x + ... + v_1 \sin x + b_2 \sin 2x + ...$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0.dx + \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{1}{\pi}\left[x.\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\cos n\pi}{n^2}\right)_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = -\frac{2}{n^2\pi} \text{ when } n \text{ is odd}$$

$$= 0, \text{ when } n \text{ is even.}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right] = \frac{(-1)^{n+1}}{n}$$

Substituting the values of $a_0, a_1, a_2 ... b_1, b_2, ...$ in (1), we get

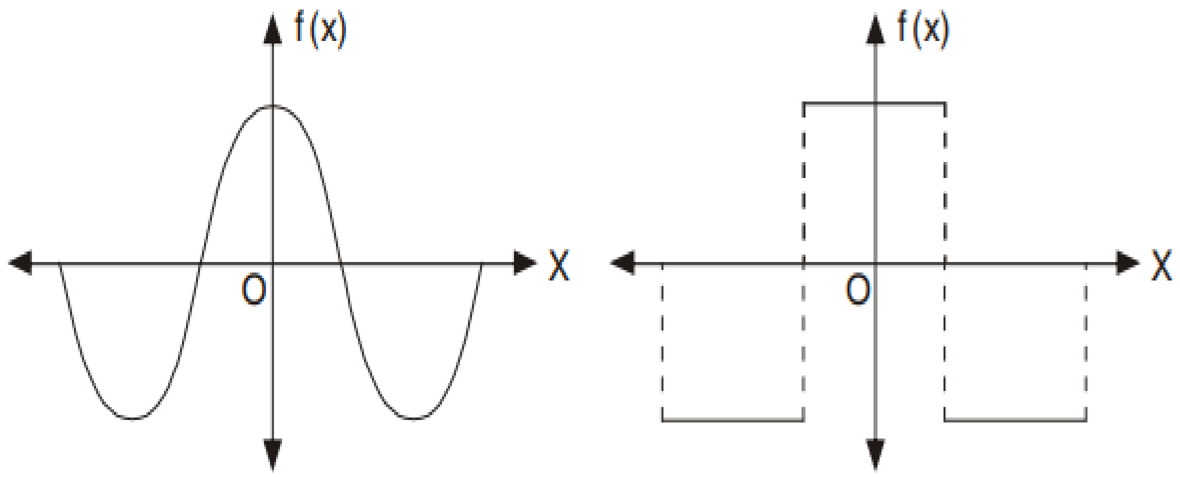
$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} ...\right] + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + ...\right]$$ **Ans.**

Symmetry Consideration

(a) Even Function

A function f (x) is said to be even (or symmetric) function if, $f(-x) = f(x)$

The graph of such a function is symmetric with respect to y-axis [f(x) axis]. Here y-axis is a mirror for the reflection of the curve.



Expansion of an Even Function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,dx$$

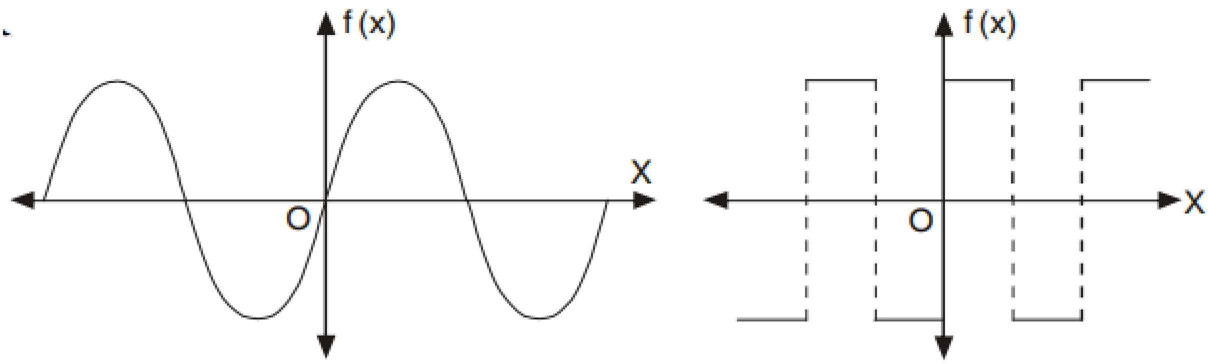
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = 0$$

- As $f(x)$ and $cos(nx)$ are both even functions.
- The product of $f(x).cos(nx)$ is also an even function.
- $sin(nx)$ is an odd function.
- The series of the even function will contain only cosine terms.

(b) Odd Function

A function $f\ (x)$ is called odd (or skew symmetric) function if

$$f(-x) = -f(x)$$



Expansion of an Odd Function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx$$

Example . *Find the Fourier series expansion of the periodic function of period* 2π

$$f(x) = x^2, \; -\pi \le x \le \pi$$

Solution. $f(x) = x^2, \; -\pi \le x \le \pi$

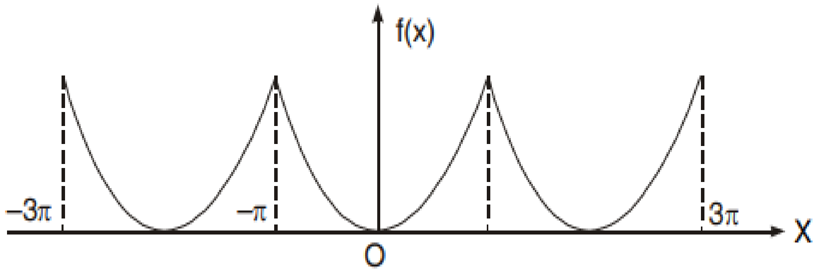
This is an even function. $\therefore \quad b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right] = \frac{4(-1)^n}{n^2}$$



Fourier series is $$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \ldots + a_n\cos nx + \ldots$$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^3} - \frac{\cos 4x}{4^2} + \ldots\right]$$

Example . *Obtain a Fourier expression for*

$$f(x) = x^3 \quad for\ -\pi < x < \pi.$$

Solution. $f(x) = x^3$ is an odd function.

$$\therefore \qquad a_0 = 0 \text{ and } a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx$$

$$\left[\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

$$= \frac{2}{\pi}\left[x^3\left(\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} + \frac{6\pi\cos n\pi}{n^3}\right] = 2.(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$x^3 = 2\left[-\left(\frac{\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(-\frac{\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(-\frac{\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x \ldots\right] \qquad \textbf{Ans.}$$

Exercise:- Find the Fourier series of wave form $f(x)$ with period 2π

A

Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi\ . \end{cases}$$

B

Let $f(x)$ be a function of period 2π such that

$$f(x) = \frac{x}{2} \text{ over the interval } 0 < x < 2\pi.$$

C

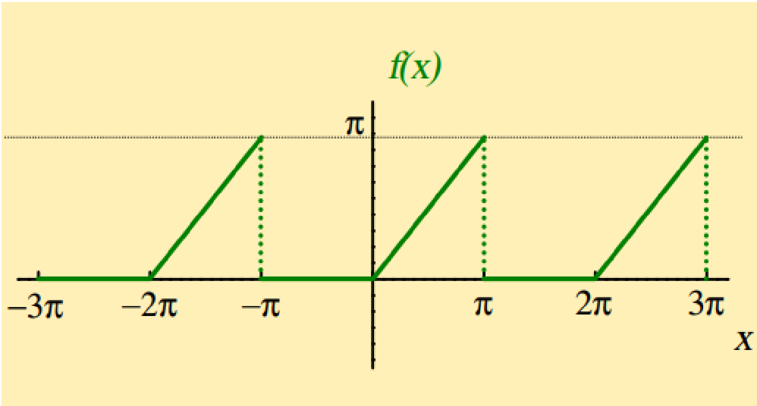
Let $f(x)$ be a function of period 2π such that

$$f(x) = \begin{cases} \pi - x, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$$

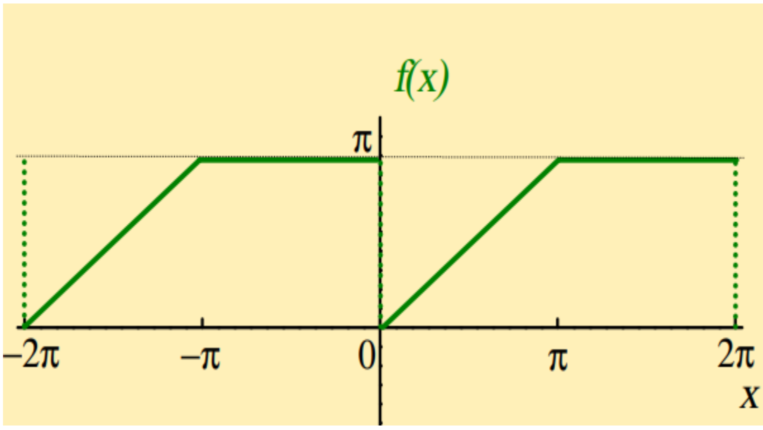
D

Given that $f(x) = x + x^2$ for $-\pi < x < \pi$

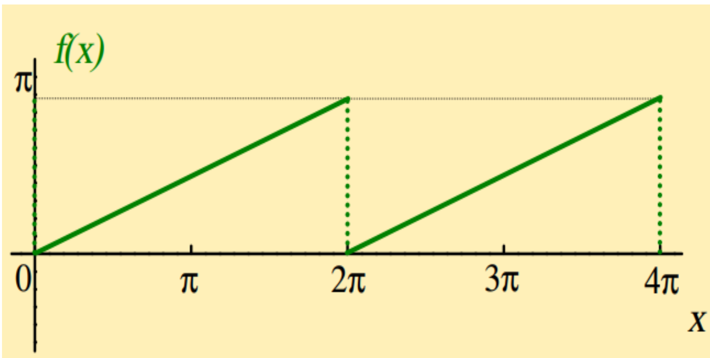
E



F



G



Fourier series for a waveform f(t) with period $T = \frac{2\pi}{w_o}$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nw_o t) + \sum_{n=1}^{\infty} b_n \sin(nw_o t)$$

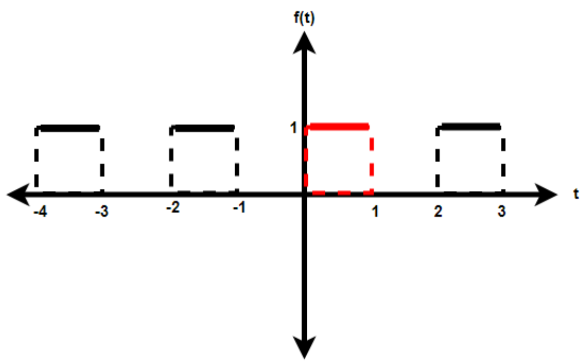
$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(nw_o t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(nw_o t)\,dt$$

$w_o = \frac{2\pi}{T} = 2\pi f$ **called fundamental frequency in radian per second**

Example:-Determine the Fourier series of wave form show in figure below.



SOLUTION

$$f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & 1 \le t \le 2 \end{cases}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nw_o t) + \sum_{n=1}^{\infty} b_n \sin(nw_o t)$$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dx$$

$$a_0 = \frac{2}{2}\left[\int_0^1 1\,dt + \int_1^2 (0)\,dx\right]$$

$$a_0 = [\mathrm{t}]_0^1 = 1$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(nw_o t)\,dt = \frac{2}{2}\left[\int_0^1 1\cos(nw_o t)\,dt + \int_1^2 0\cos(nw_o t)\,dt\right]$$

$$a_n = \frac{\sin(nw_o t)]_0^1}{nw_o} = \frac{\sin\left(n\frac{2\pi}{T}t\right)\Big]_0^1}{n\frac{2\pi}{T}} = \frac{\sin\left(n\frac{2\pi}{2}t\right)\Big]_0^1}{n\frac{2\pi}{2}} = \frac{1}{n\pi}\sin(n\pi t)]_0^1$$

$$a_n = \frac{1}{n\pi}[\sin(n\pi) - \sin(0)] = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(nw_o t)\,dt = \frac{2}{2}\left[\int_0^1 1\sin(nw_o t)\,dt + \int_1^2 0\sin(nw_o t)\,dt\right]$$

$$b_n = -\frac{\cos(nw_o t)]_0^1}{nw_o} = -\frac{\cos\left(n\frac{2\pi}{T}t\right)\Big]_0^1}{n\frac{2\pi}{T}} = -\frac{\cos\left(n\frac{2\pi}{2}t\right)\Big]_0^1}{n\frac{2\pi}{2}} = -\frac{1}{n\pi}\cos(n\pi t)]_0^1$$

$$b_n = -\frac{1}{n\pi}[\cos(n\pi) - 1] = \frac{1}{n\pi}[1 - \cos(n\pi)]$$

$$b_n = \frac{1}{n\pi}[1 - (-1)^n] = \begin{cases}\frac{1}{n\pi}, & n = odd\\ 0, & n = even\end{cases}$$

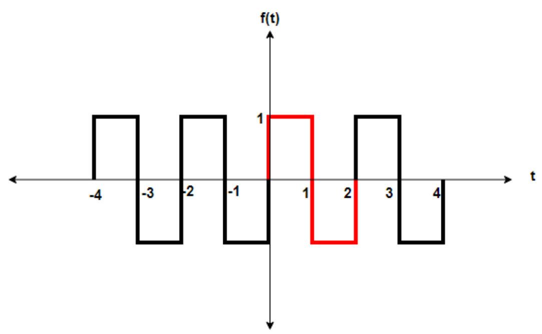
$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} 0\cos(nw_o t) + \sum_{n=1}^{\infty} b_n\sin(n\pi t)$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\pi t) + \frac{2}{3\pi}\sin(3\pi t) + \frac{2}{5\pi}\sin(5\pi t) + \cdots$$

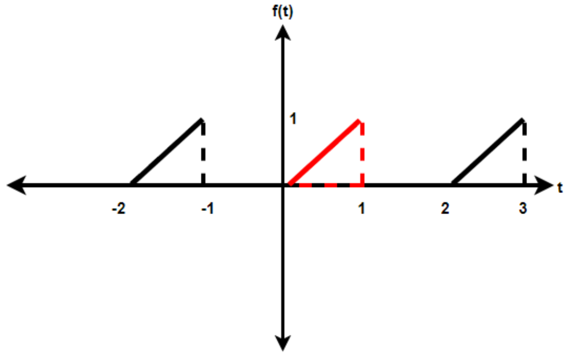
$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi t) \quad when\ n = 1,3,5,7\ or\ n = 2k-1 \quad k \geq 1$$

Exercise:-Determine the Fourier series of wave form show in figure below.

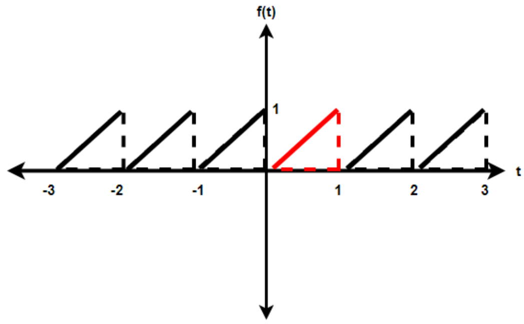
A



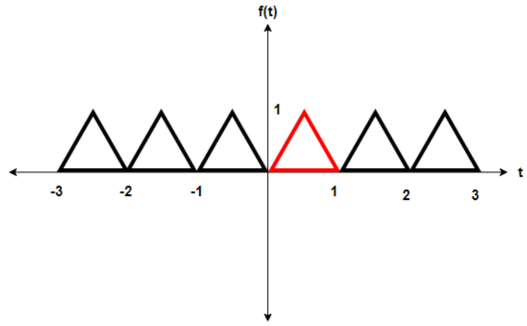
B


f(t)
1
-2
-1
1
2
3
t

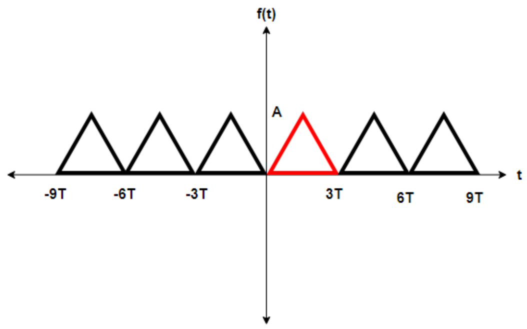
C


f(t)
1
-3
-2
-1
1
2
3
t

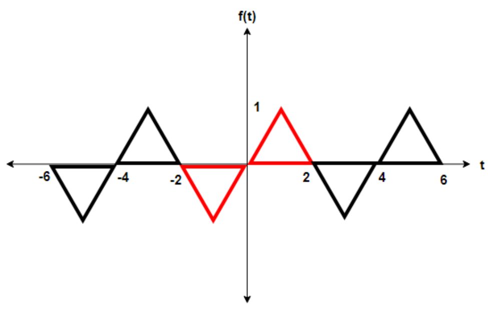
D


f(t)
1
-3
-2
-1
1
2
3
t

E


f(t)
A
-9T
-6T
-3T
3T
6T
9T
t

F


f(t)
1
-6
-4
-2
2
4
6
t