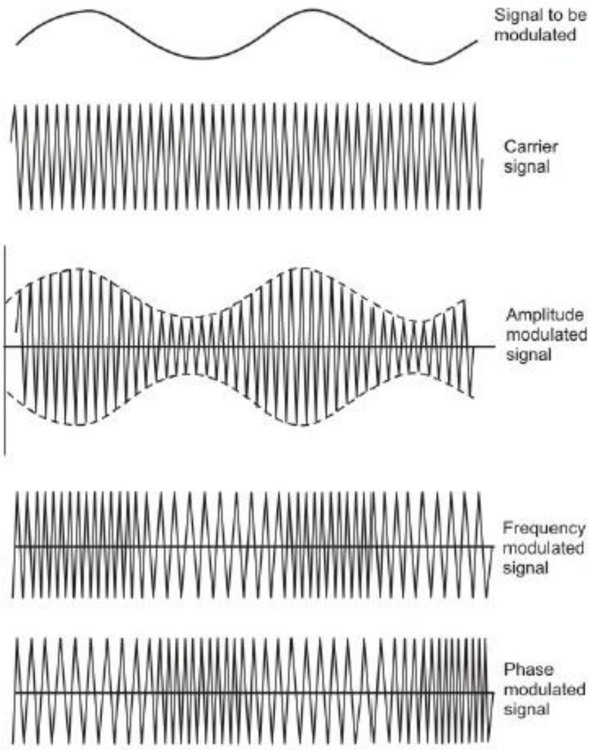
Angle Modulation

In AM signals, the amplitude of a carrier is modulated by a signal $m(t)$, and, hence, the information content of $m(t)$ is in the amplitude variations of the carrier. As we have seen, the other two parameters of the carrier sinusoid, namely its frequency and phase, can also be varied in proportion to the message signal as frequency-modulated and phase-modulated signals, respectively. We now describe the essence of frequency modulation (FM) and phase modulation (PM).



Analog data to Analog signal modulation techniques

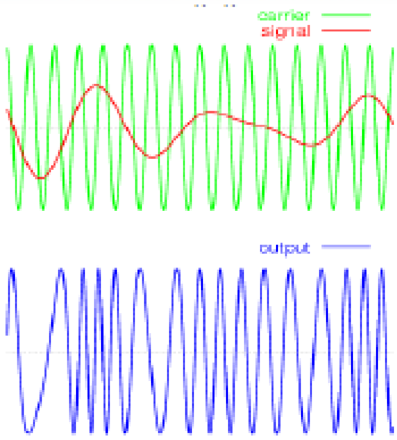


Figure 2.5.12 Angle modulation

Broadcasting was in its infancy. However, there was an active search for techniques to reduce noise (static). Since the noise power is proportional to the modulated signal bandwidth (sidebands), efforts were focused on finding a modulation scheme that would reduce the bandwidth. More important still, bandwidth reduction also allows more users, and there were rumors of a new method that had been discovered for eliminating sidebands (no sidebands, no bandwidth !). The idea of **frequency modulation (FM),** where the carrier frequency would be varied in proportion to the message $m(t)$, was quite intriguing.

The Concept of Instantaneous Frequency

The carrier angular frequency $w(t)$ would be varied with time

$$\boldsymbol{w(t) = w_c + km(t)}$$

k is an arbitrary constant.

If the peak amplitude of $\mathbf{m(t)}$ is $\mathbf{m_p}$

then the maximum values of the carrier frequency would be $\boldsymbol{w_c + km_p}$ minimum values $\boldsymbol{w_c - km_p}$

the spectral components would remain within this band with a bandwidth $\mathbf{2km_p}$ centered at quite intriguing.

Let us consider a generalized sinusoidal signal $\boldsymbol{\varphi}(\mathbf{t})$ given by

$$\boldsymbol{\varphi}(\mathbf{t}) = \boldsymbol{A\cos\theta(t)}$$

where $\theta(t)$ is the generalized angle and is a function of t. Figure below shows a hypothetical case of $\theta(t)$. The generalized angle for a conventional sinusoid $A\cos(w_c t + \theta_o)$ is a straight line $(w_c t + \theta_o)$, as shown in figure. A hypothetical case general angle of $\theta(t)$happens to be tangential to the angle $(w_c t + \theta_o)$, at some instant t. The crucial point is that, around t, over a small interval $\Delta t \rightarrow 0$, the signal $\varphi(t) = A\cos\theta(t)$ and the sinusoid $A\cos(w_c t + \theta_o)$ are identical; that is,

$$\boldsymbol{\varphi(t) = A\cos(w_c t + \theta_o)} \qquad \boldsymbol{t_1} < t < \boldsymbol{t_2}$$

We are certainly justified in saying that over this small interval Δt the angular frequency of $\varphi(t)$ is w_c - Because $(w_c t + \theta_o)$ is tangential to $\theta(t)$, the angular frequency of $\varphi(t)$ is the slope of its angle $\theta(t)$ over this small interval. We can generalize this concept at **every instant** and define that the **instantaneous frequency** $\boldsymbol{w_i}$ at any instant t is the slope of $\theta(t)$ at t.

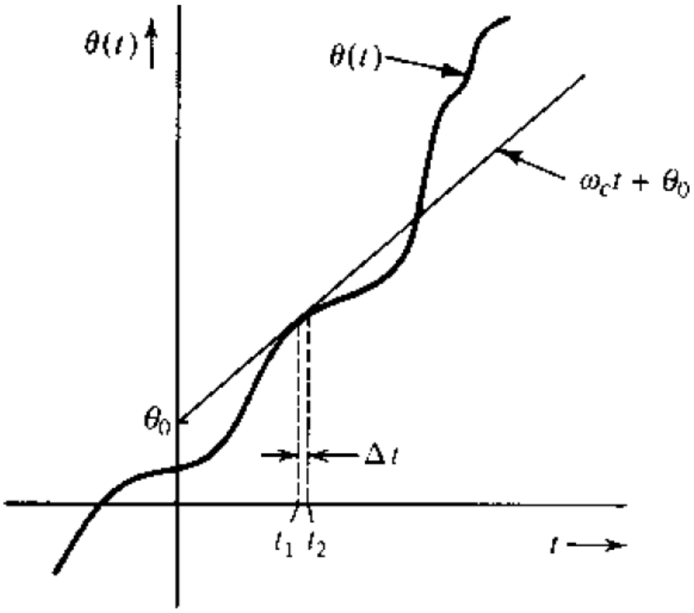


Figure: Concept of instantaneous frequency.

$$\boldsymbol{w_i} = \frac{\mathbf{d\theta}}{\mathbf{dt}}$$

$$\boldsymbol{\theta} = \int_{-\infty}^{t} \boldsymbol{w_i}(\propto)\boldsymbol{d} \propto$$

Now we can see the possibility of transmitting the information of $m(t)$ by varying the angle θ of a carrier. Such techniques of modulation, where the angle of the carrier is varied in some manner with a modulating signal $m(t)$, are known as ***angle modulation*** or ***exponential modulation.*** Two simple possibilities are phase modulation (PM) and frequency modulation (FM) .

In PM, the angle $\boldsymbol{\theta(t)}$ is varied linearly with $\mathbf{m(t)}$:

$$\boldsymbol{\theta(t) = w_c t + \theta_o + k_p m(t)}$$

where $\boldsymbol{k_p}$ is a constant and $\boldsymbol{w_c}$ is the carrier frequency. Assuming $\boldsymbol{\theta_o = 0}$

$$\boldsymbol{\varphi_{PM}(t) = A\,cos(w_c t + k_p m(t))}$$

The instantaneous angular frequency $w_i(t)$ in this case is given by

$$\mathbf{w_i(t) = \frac{d\theta}{dt} = w_c + k_p \dot{m}(t)}$$

$$\mathbf{f_i(t) = \frac{d\theta}{dt} = f_c + \frac{k_p}{2\pi} \dot{m}(t)}$$

In FM the instantaneous angular frequency w_i is

$$w_i(t) = w_c + k_f m(t)$$

$$f_i(t) = f_c + \frac{k_f}{2\pi} m(t)$$

where k_f is a constant. The angle θ(t) is now

$$\theta = \int_{-\infty}^{t} w_i(\propto) d\propto = w_c t + k_f \int_{-\infty}^{t} m(\propto) d\propto$$

$$\varphi_{FM}(t) = A\cos(w_c t + k_f \int_{-\infty}^{t} m(\propto) d\propto)$$

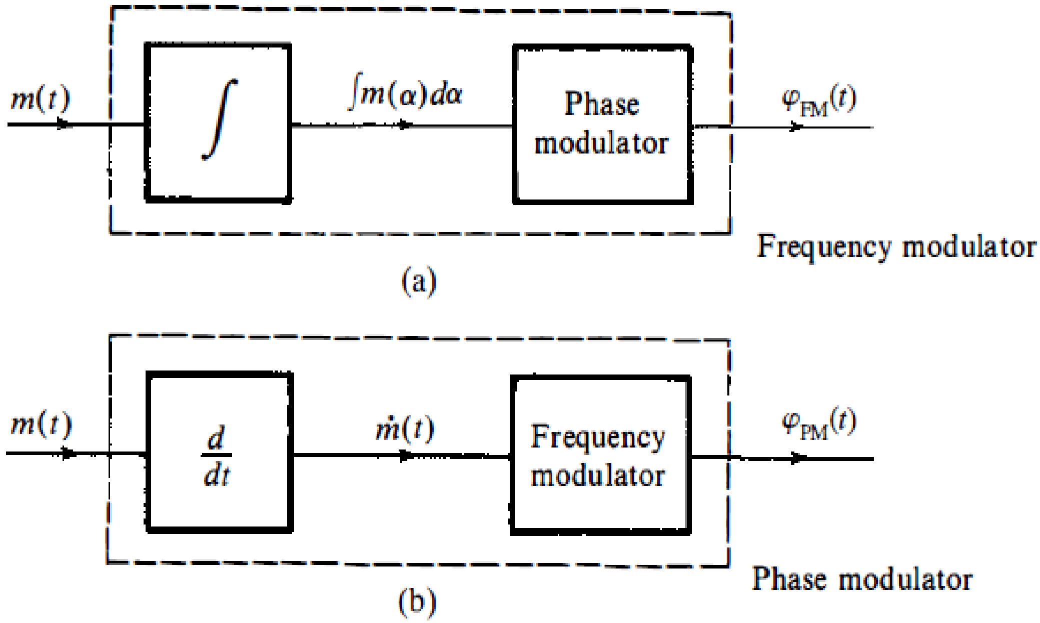


Figure: Phase and Frequency modulation are equivalent and interchangeable.

Power of Angle-Modulated Wave

Although the instantaneous frequency and phase of an angle-modulated wave can vary with time, the amplitude A remains constant. Hence, the power of an angle-modulated wave (PM or FM) is always $A^2/2$, regardless of the value of k_f or k_p.

$$P = \frac{A^2}{2}$$

Bandwidth of Angle-Modulate Waves

Unlike AM, angle modulation is nonlinear and no properties o f Fourier transform can be directly applied for its bandwidth analysis. To determine the bandwidth of an FM wave, let us define.

$$a(t) = \int_{-\infty}^{t} m(\alpha) d\alpha$$

$$\varphi_{FM}(t) = A\cos(w_c t + k_f a(t))$$

$$\dot{\varphi}_{FM}(t) = A e^{j[w_c t + k_f a(t)]}$$

$$\varphi_{FM}(t) = Re[\dot{\varphi}_{FM}(t)]$$

Expanding the exponential $e^{jk_f a(t)}$ in power series yields

$$\dot{\varphi}_{FM}(t) = A\left[1 + jk_f a(t) - \frac{{k_f}^2 a(t)^2}{2!} - j\frac{{k_f}^3 a(t)^3}{3!} + \cdots \ldots + j^n \frac{{k_f}^n a(t)^n}{n!}\right] e^{jw_c t}$$

$$\varphi_{FM}(t) = Re[\dot{\varphi}_{FM}(t)] = A\left[\cos(w_c t) - k_f a(t)\sin(w_c t) - \frac{{k_f}^2 a(t)^2}{2!}\cos(w_c t) + \frac{{k_f}^3 a(t)^3}{3!}\sin(w_c t) + \cdots\right]$$

The modulated wave consists of an unmodulated carrier plus various amplitude-modulated terms, such as $a(t)\sin(w_c t)$, $a(t)^2\cos(w_c t)$, $a(t)^3\sin(w_c t)$. The signal $a(t)$ is an integral of $m(t)$**.** If $M(f)$ is band-limited to B, $A(f)$ is also band-limited to B. The spectrum of $a(t)^2$ is simply $A(f) * A(f)$ and is band-limited to $2B$. Similarly, the spectrum of $a(t)^n$ is band-limited to nB**.**

1- Narrowband Angle Modulation Approximation

When k_f is very small such that

$$|k_f a(t)| \ll 1$$

then all higher order terms are negligible except for the first two. We then have a good approximation**. Narrowband FM (NBFM)**. Similarly, the **narrowband PM (NBPM)** signal is approximated by

$$\varphi_{FM}(t) = A[\cos(w_c t) - k_f a(t)\sin(w_c t)]$$

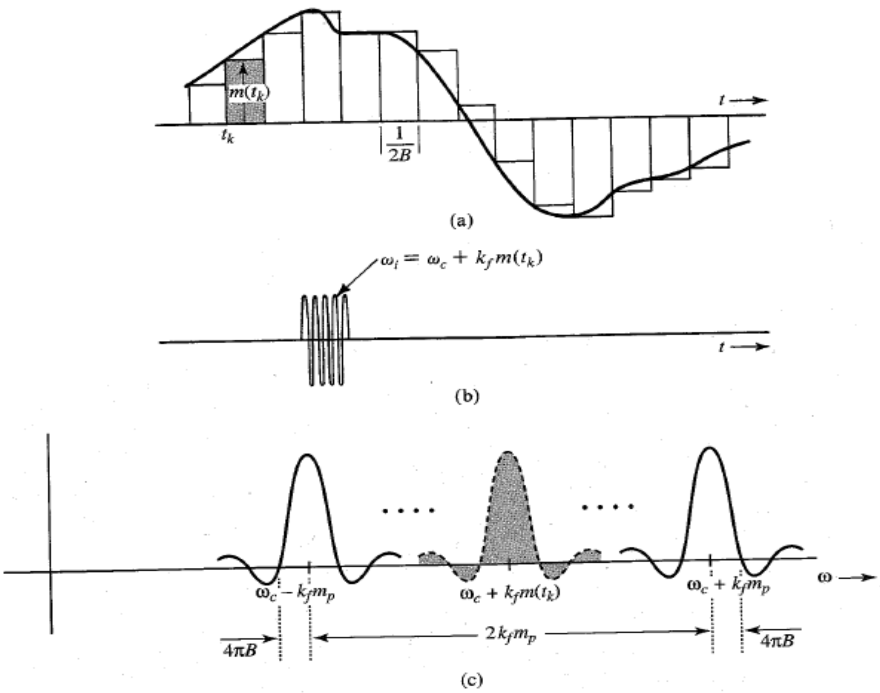
$$\varphi_{PM}(t) = A[\cos(w_c t) - k_p m(t)\sin(w_c t)]$$

NBPM also has the approximate

$$NBFM = NBPM = B.W = 2B$$

2- Wideband FM (WBFM) Bandwidth Analysis

Note that an FM signal is meaningful only if its frequency deviation is large enough. In other words, practical FM chooses the constant $\boldsymbol{k_f}$ large enough that the condition $|\boldsymbol{k_f}\mathbf{a(t)}| \ll \mathbf{1}$ is not satisfied. We call FM signals in such cases **wideband FM (WBFM).** Thus, in analyzing the bandwidth of WBFM, we cannot ignore all the higher order .



Hence, the maximum and the minimum significant frequencies in this spectrum are $\mathbf{w_c} + \boldsymbol{k_f}\mathbf{m_p} + \mathbf{4\pi B}$ and $\mathbf{w_c} - \boldsymbol{k_f}\mathbf{m_p} - \mathbf{4\pi B}$, respectively. The FM spectrum bandwidth is approximately

$$\boldsymbol{B_{Fm}} = \frac{\mathbf{1}}{\mathbf{2\pi}}\left(\mathbf{2}\boldsymbol{k_f}\mathbf{m_p} + \mathbf{8\pi B}\right) = \mathbf{2}\left(\boldsymbol{k_f}\frac{\mathbf{m_p}}{\mathbf{2\pi}} + \mathbf{2B}\right) \;\; Hz$$

we shall denote the peak **frequency deviation** in hertz by Δf . Thus,

$$\boldsymbol{\Delta f} = \boldsymbol{k_f}\frac{\mathbf{m_p}}{\mathbf{2\pi}}$$

The estimated FM bandwidth (in hertz) can then be expressed as

$$\boldsymbol{B_{Fm}} \approx \mathbf{2}(\boldsymbol{\Delta f} + \mathbf{2B}) \;\; Hz$$

The bandwidth estimate thus obtained is somewhat higher than the actual value because this is the bandwidth corresponding to the staircase approximation of $m(t)$, not the actual $m(t)$, which is considerably smoother. Hence, the actual FM bandwidth is somewhat smaller than this value.

we observe that for the case of NBFM, $\boldsymbol{k_f}$is very small. Hence, given a fixed $\mathbf{m_p}$, $\Delta \boldsymbol{f}$ is very small.

$$\boldsymbol{B_{Fm}} \approx \mathbf{4}\boldsymbol{B} \quad Hz$$

But we showed earlier that for narrowband, the FM bandwidth is approximately $\mathbf{2}\boldsymbol{B}\ \boldsymbol{Hz}$. This indicates that a better bandwidth estimate is

$$\boldsymbol{B_{FM}} \approx \mathbf{2}(\Delta \boldsymbol{f} + \mathbf{B}) \quad Hz \qquad \textbf{Carson's rule}$$

This formula goes under the name **Carson's rule**

Observe that for a truly wideband case, where $\Delta \boldsymbol{f} \gg \mathbf{B}$

$$\boldsymbol{B_{Fm}} \approx \mathbf{2}\Delta \boldsymbol{f} \quad \boldsymbol{Hz}$$

We define a **deviation ratio** $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta} = \frac{\Delta \boldsymbol{f}}{\boldsymbol{B}} = \boldsymbol{k_f}\frac{\mathbf{m_p}}{\mathbf{2\pi}\boldsymbol{B}}$$

Carson's rule can be expressed in terms of the **deviation ratio** as

$$\boldsymbol{B_{Fm}} \approx \mathbf{2}\boldsymbol{B}(\boldsymbol{\beta} + \mathbf{1}) \quad Hz$$

The **deviation ratio** controls the amount of modulation and, consequently, plays a role similar to the **modulation index in AM**. Indeed, for the special case of **tone-modulated FM**, the deviation ratio $\boldsymbol{\beta}$ is called the **modulation index.**

Phase Modulation(PM)

All the results derived for FM can be directly applied to PM. Thus, for PM, the instantaneous frequency is given by

$$\mathrm{w_i(t) = w_c + k_p \dot{m}(t)}$$

$$\mathbf{f_i(t) = f_c + \frac{k_p}{2\pi}\dot{m}(t)}$$

Therefore, the peak frequency deviation $\Delta \boldsymbol{f}$ is given by

$$\Delta \boldsymbol{f} = \boldsymbol{k_p}\frac{\mathbf{\dot{m}_p}}{\mathbf{2\pi}}$$

$$\boldsymbol{B_{PM}} = \mathbf{2}(\Delta \boldsymbol{f} + \mathbf{B}) \quad Hz$$