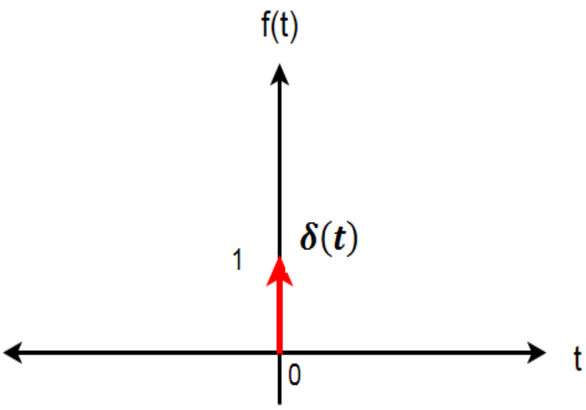
Singularity Function Continuous-Time Signal

1- Unit Impulse Function $\delta(t)$

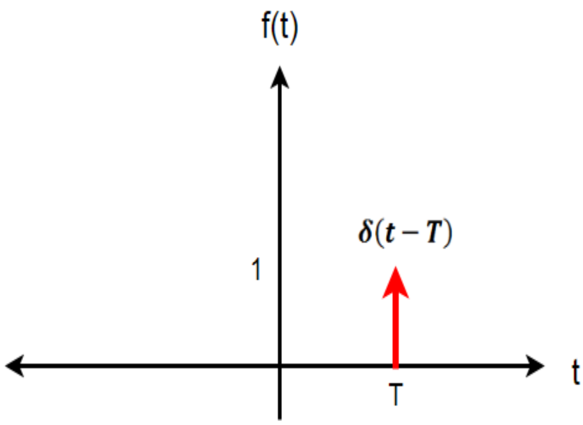
The unit impulse function $\delta(t)$ is one of the most important functions in the study of signals and systems. Its definition and application provide much convenience that is not permissible in pure mathematics. The unit impulse function define:

$$\delta(t) = 0, \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



Unit Impulse $\delta(t)$



Unit Impulse $\delta(t-T)$

Example: Sketch the signals $f(t)$

1- $f(t) = \delta(t)$

2- $f(t) = -\delta(t+3)$

3- $f(t) = 2\delta(t-1)$

4- $f(t) = 2\delta(t-4) - \delta(t+5)$

5- $f(t) = A\delta(t+T)$ **(H.W)**

6- $f(t) = \int_{-\infty}^{\infty} \delta(t-3)\, dt = 1$ **(H.W)**

Multiplication of a Function by an Impulse

Let us now consider what happens when we multiply the unit impulse $\delta(t)$ by a function $\phi(t)$ that is known to be continuous at $t = 0$. Since the impulse exists only at $t = 0$, and the value of $\phi(t)$ at $t = 0$ is $\phi(0)$, we obtain

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

Example: Find the signals $f(t)$

1- $f(t) = \phi(t)\delta(t+T)$
2- $f(t) = e^{t}\delta(t-3)$
3- $f(t) = \int_{-\infty}^{\infty} \phi(t)\delta(t)\, dt$
4- $f(t) = \int_{-\infty}^{\infty} e^{t}\delta(t)\, dt$
5- $f(t) = \int_{-\infty}^{\infty} e^{t}\delta(t-1)\, dt$ $(\boldsymbol{H.W})$
6- $f(t) = \int_{-\infty}^{\infty} e^{2+t}\delta(t+2)\, dt$
7- $f(t) = \int_{-\infty}^{-5} (t^2+1)\delta(t+3)\, dt$ $(\boldsymbol{H.W})$

2- Unit Step Function $u(t)$

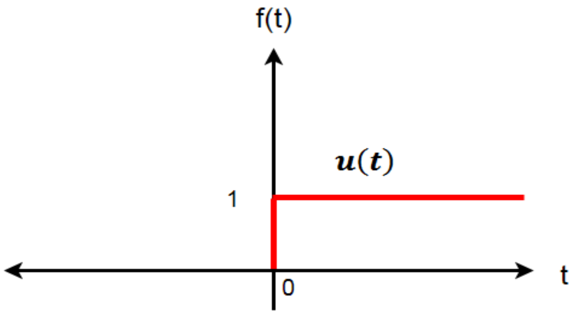
Another familiar and useful function is the unit step function $u(t)$often encountered in circuit analysis and defined by

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

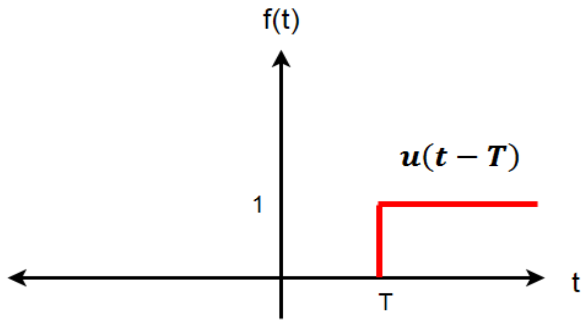
$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} = u(t)$$

From this result it follows that

$$\frac{du(t)}{dt} = \delta(\tau)$$



Unit Step Function $u(t)$



Unit Step Function $u(t-T)$

Example: Sketch the signals $f(t)$.

1- $f(t) = u(t-1)$

2- $f(t) = 2u(t+3)$

3- $f(t) = -5u(t+1)$ **(H.W)**

4- $f(t) = u(t-1) - u(t-3)$

5- $f(t) = u(t) + u(t-3)$ **(H.W)**

6- $f(t) = u(-t)$

7- $f(t) = u(t) - u(-t) = sgn(t)$

8- $f(t) = e^{-t}u(t)$

9- $f(t) = e^{-t}sgn(t)$ $(\boldsymbol{H.W})$

Fourier Transform

Analysis of a non-periodic function over entire interval

A non periodic signal may be assumed as a limiting case of a periodic signal where the signal approaches infinity. We can use this approach to develop the frequency domain of a non periodic signal over an entire interval

- **The Fourier transform of a finite duration signal can be found using the formula**

$$\boldsymbol{F(w) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-jwt} dt}$$

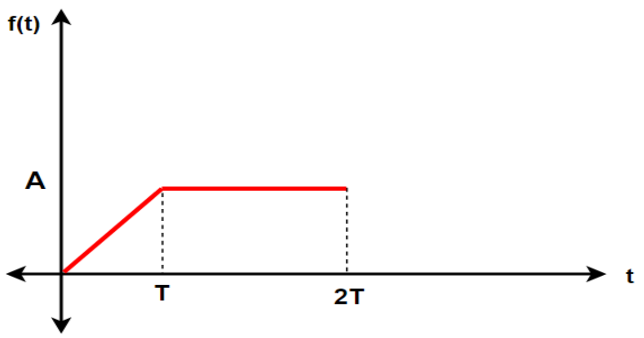
- **The inverse Fourier transform is given by**

$$f(t) = \mathcal{F}^{-1}[F(w)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\, e^{jwt} dw$$

$$f(t) \rightleftharpoons F(w)$$

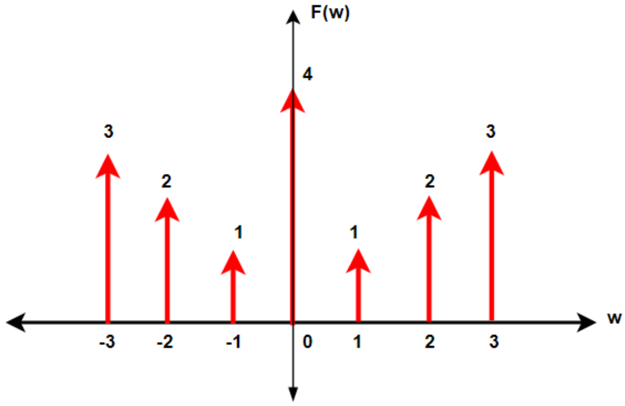
Example: Find the Fourier transform of

1- $f(t) = \delta(t)$
2- $f(t) = \delta(t-2)$
3- $f(t) = -4\delta(t+2) + 5\delta(t) - 2\delta(t-1) + \delta(t-4)$ $(\boldsymbol{H.W})$
4- $f(t) = \delta(t-A) \quad A < 0$ $(\boldsymbol{H.W})$
5- $f(t) = e^{jw_o t}$
6- $f(t) = \cos w_o t$
7- $f(t) = \sin w_o t$ $(\boldsymbol{H.W})$
8- $f(t) = u(t)$
9- $f(t) = u(t+1)$ $(\boldsymbol{H.W})$
10- $f(t) = e^{-at}u(t)$
11- $f(t) = e^{at}u(-t)$ $(\boldsymbol{H.W})$
12- $f(t) = e^{-a|t|}sgn(t)$$(\boldsymbol{H.W})$
13- $f(t) = e^{-at}\delta(t)$ $(\boldsymbol{H.W})$
14- $f(t) = \begin{cases} 10 & 0 \le t \le 2 \\ 0 & elsewhere \end{cases}$
15- $f(t) = u(t) - u(t+4)$$(\boldsymbol{H.W})$
16- Find the Fourier transform of the signal $f(t)$ shown in below figure: $(\boldsymbol{H.W})$



Example: Find the inverse Fourier transform of

1- $F(w) = \delta(w)$
2- $F(w) = \delta(w - w_o)$
3- $F(w) = 1$ **(H.W)**
4- Find the inverse Fourier transform of the signal $F(w)$ shown in below figure **(H.W)**



Frequency Shift Property

Frequency shifting property states that the multiplication of a function $f(t)$ by $e^{jw_o t}$ equivalent to shifting its Fourier transform $F(w)$ in the positive direction by an amount w_o, This mean the spectrum $F(w)$ is translated by an amount w_o.

If $\quad \boldsymbol{f(t) \rightleftharpoons F(w)}$

Then $\quad \boldsymbol{e^{jw_o t} f(t) \rightleftharpoons F(w - w_o)}$

Proof

$$F(w) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-jwt} dt$$

$$\mathcal{F}\left[e^{jw_o t} f(t)\right] = \int_{-\infty}^{\infty} e^{jw_o t} f(t)\, e^{-jwt} dt$$

$$\mathcal{F}\left[e^{jw_o t} f(t)\right] = \int_{-\infty}^{\infty} f(t)\, e^{-j(w-w_o)t} dt$$

$$\mathcal{F}\left[e^{jw_o t} f(t)\right] = F(w - w_o)$$

Example

1- $f(t) = \mathrm{x(t)} \cos w_o t$
2- $f(t) = 2\,\mathrm{x(t)}\, e^{-jw_o t}$ **(H.W)**