PROBABILITY, SIGNALS & SYSTEMS

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DISCRETE FOURIER TRANSFORM

Discrete Fourier Transform

continuous $$X(F) = \int_{-\infty}^{\infty} x(t)\ e^{-j2\pi Ft} dt$$

discrete $$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{j2\pi kn}{N}}$$

k-th frequency

evaluating at n of N samples

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DISCRETE FOURIER TRANSFORM

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{j2\pi kn}{N}}$$

$\frac{j2\pi kn}{N} \rightarrow b_n$

"kth" frequency bin

$$X_k = x_0 e^{-b_0 j} + x_1 e^{-b_1 j} + \ldots + x_n e^{-b_{N-1} j}$$

"nth" sample value

Euler's Formula:

$$e^{jx} = \cos x + j \sin x$$

$$X_k = x_0[\cos(-b_0) + j\sin(-b_0)] + \cdots$$

$$X_k = A_k + B_k j$$

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EXAMPLE

Find the 4-Point DFT of the sequence $x(n) = \cos\frac{n\pi}{4}$.

Solution:

Given N =4

x(n) = $\{\cos(0), \cos(\frac{\pi}{4}), \cos(\frac{\pi}{2}), \cos(\frac{3\pi}{4})\}$ = **{1, 0.707, 0, -0.707}**

The N-point DFT of the sequence x(n) is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}, \quad k = 0, 1, 2, \dots, N-1$$

$$X(k) = \sum_{n=0}^{3} x(n)e^{-j2\pi nk/4}, \quad k = 0, 1, 2, 3$$

$$= \sum_{n=0}^{3} x(n)e^{-j\pi nk/2}, k = 0, 1, 2, 3$$

For k = 0

$X(0) = \sum_{n=0}^{3} x(n)e^{-j\pi(0)n/2} = \sum_{n=0}^{3} x(n) = \mathbf{1}$

For k = 1

$X(1) = \sum_{n=0}^{3} x(n)e^{-j\pi(1)n/2} = 1 + 0.707e^{-j\pi/2} + 0 + (-0.707)e^{-j3\pi/2}$

$= 1 + 0.707(-j) + 0 - (0.707)(j) = \mathbf{1 - j\,1.414}$

EXAMPLE

For k = 2

$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi(2)n/2} = 1 + 0.707e^{-j\pi} + 0 + (-0.707)e^{-j3\pi}$

$= 1 + 0.707(-1) + 0 - (0.707)(-1) = \mathbf{1}$

For k = 3

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j\pi(3)n/2} = 1 + 0.707e^{-j3\pi/2} + 0 + (-0.707)e^{-j9\pi/2}$$

$$= 1 + 0.707(j) + 0 - (0.707)(-j) = \mathbf{1 + j\,1.414}$$

X(k) = {1, $\mathbf{1 - j\,1.414}$, 1, $\mathbf{1 + j\,1.414}$}

EXAMPLE

Find the inverse DFT of $X(k) = \{1, 2, 3, 4\}$.

Solution:

The IDFT is given by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N}, \quad n = 0, 1, 2, \dots, N-1$$

Given N = 4, $x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j2\pi nk/4}, \quad n = 0, 1, 2, 3$

When n = 0

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j2\pi(0)k/4} = \frac{1}{4}(1 + 2 + 3 + 4) = \mathbf{5/2}$$

When n = 1

$$X(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j2\pi(1)k/4} = \frac{1}{4}\left(1 + 2e^{j\pi/2} + 3e^{j\pi} + 4e^{j3\pi/2}\right) =$$

$$\frac{1}{4}(1 + 2(j) + 3(-1) + 4(-j)) = \frac{1}{4}(2 - 2j) = \frac{\mathbf{1}}{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{j}$$

EXAMPLE

When n = 2

$X(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{\,j2\pi(2)k/4} = \frac{1}{4}\left(1 + 2e^{j\pi} + 3e^{j2\pi} + 4e^{j3\pi}\right) =$

$$\frac{1}{4}(1 + 2(-1) + 3(1) + 4(-1)) = \frac{1}{4}\,(-2) = -\frac{\mathbf{1}}{\mathbf{2}}$$

When n = 3

$X(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{\,j2\pi(3)k/4} = \frac{1}{4}\left(1 + 2e^{j3\pi/2} + 3e^{j3\pi} + 4e^{j9\pi/2}\right) =$

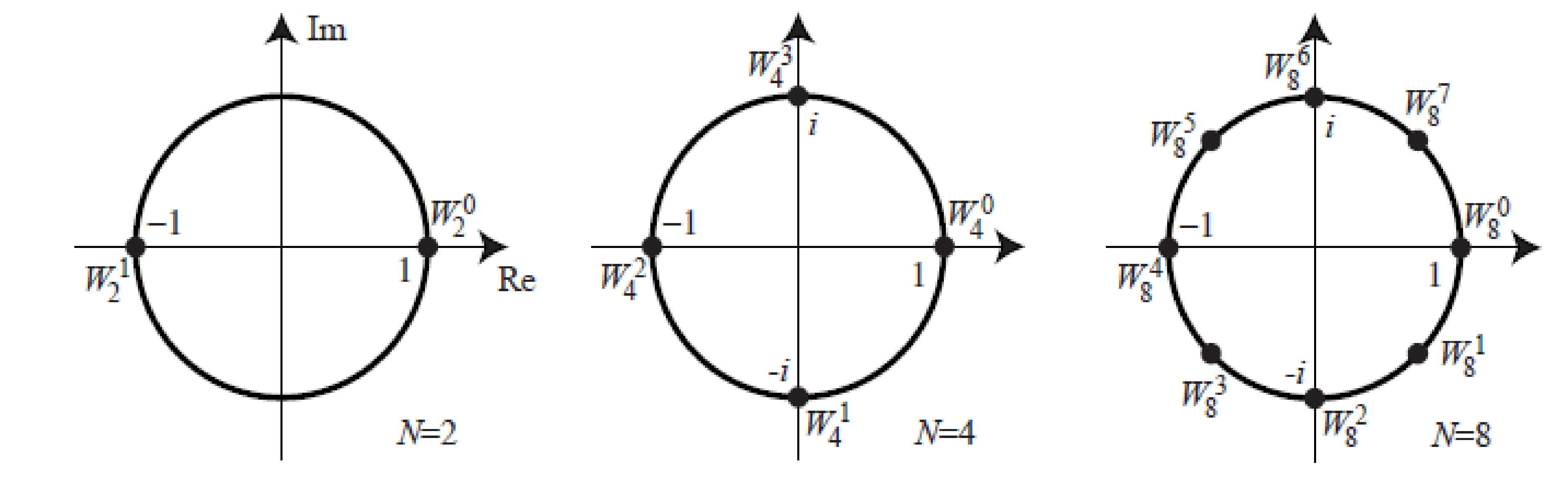
$$\frac{1}{4}(1 + 2(-j) + 3(-1) + 4(j)) = -\frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{j}$$

X(n) = $\{\frac{5}{2}, \frac{1}{2} - \frac{1}{2}\,j\,, -\frac{1}{2}\,, -\frac{1}{2} + \frac{1}{2}j\}$

FAST FOURIER TRANSFORM (FFT) ALGORITHM

- The FFT is a fast algorithm for computing the DFT. If we take the 2-point DFT and 4-point DFT and generalize them to 8-point, 16-point, ..., 2*r*-point, we get the FFT algorithm.
- The equation DFT
- $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}, \quad k = 0, 1, 2, \ldots, N-1$
- Then $X(k) = \sum_{n=0}^{N-1} x(n)\mathrm{w}^{nk}{}_{N}, \quad k = 0, 1, 2, \ldots, N-1$
- where $\mathbf{w}^{k}{}_{N} = e^{-i2\pi k/N}$,$\mathrm{N} = 2,4,8,16,\ldots\ldots.2^{n}$

FAST FOURIER TRANSFORM (FFT) ALGORITHM



$W_8^{\,0}=1+i0$ $W_8^{\,1}=0.707-i0.707$ $W_8^{\,2}=0-i=-i$

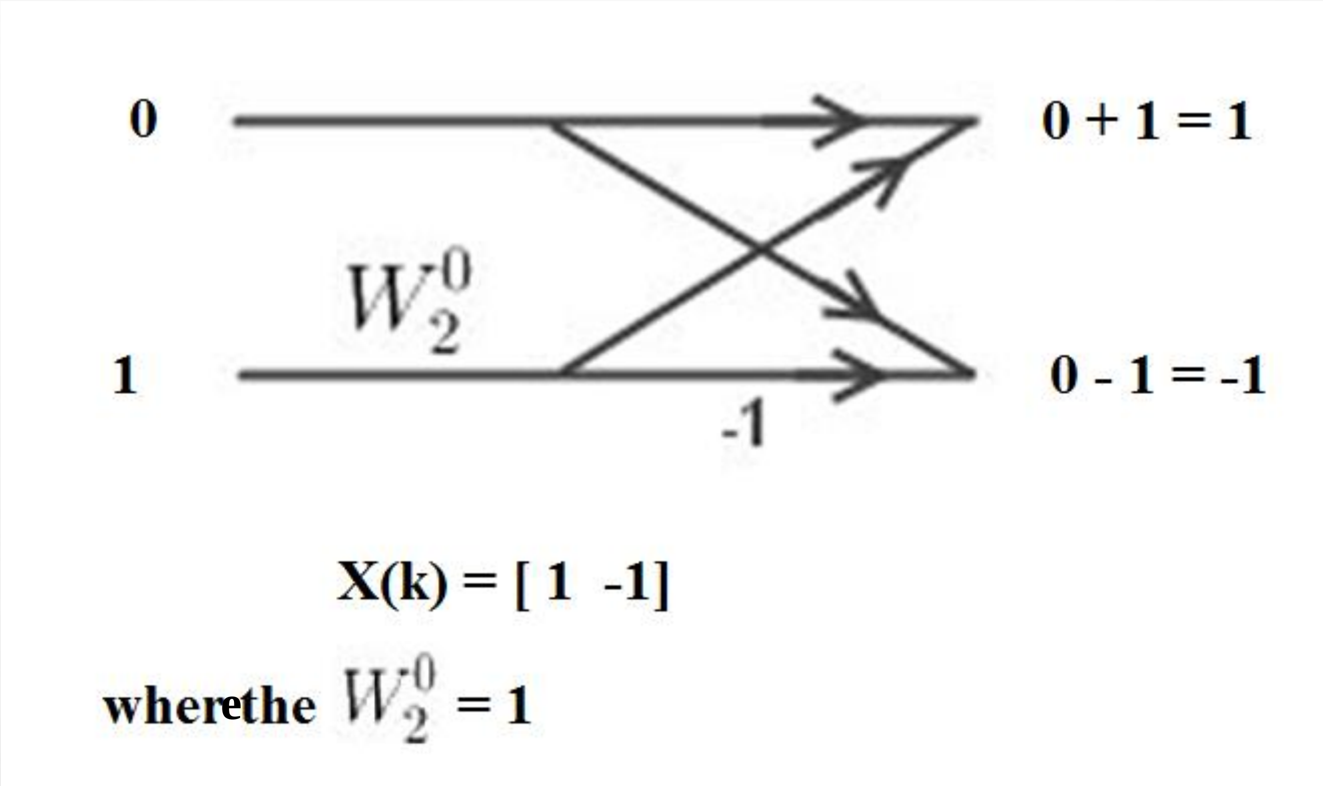
$W_8^{\,3}= -0.707-i0.707$ $W_8^{\,4}= -1-i0 =-1$ $W_8^{\,5}=- 0.707+i0.707$

$W_8^{\,6}=0+i1=i$ $W_8^{\,7}=0.707+i0.707$

HOW TO CREATE A 2 POINT INPUT BUTTERFLY

Example: find the FFT for the signal X(n)

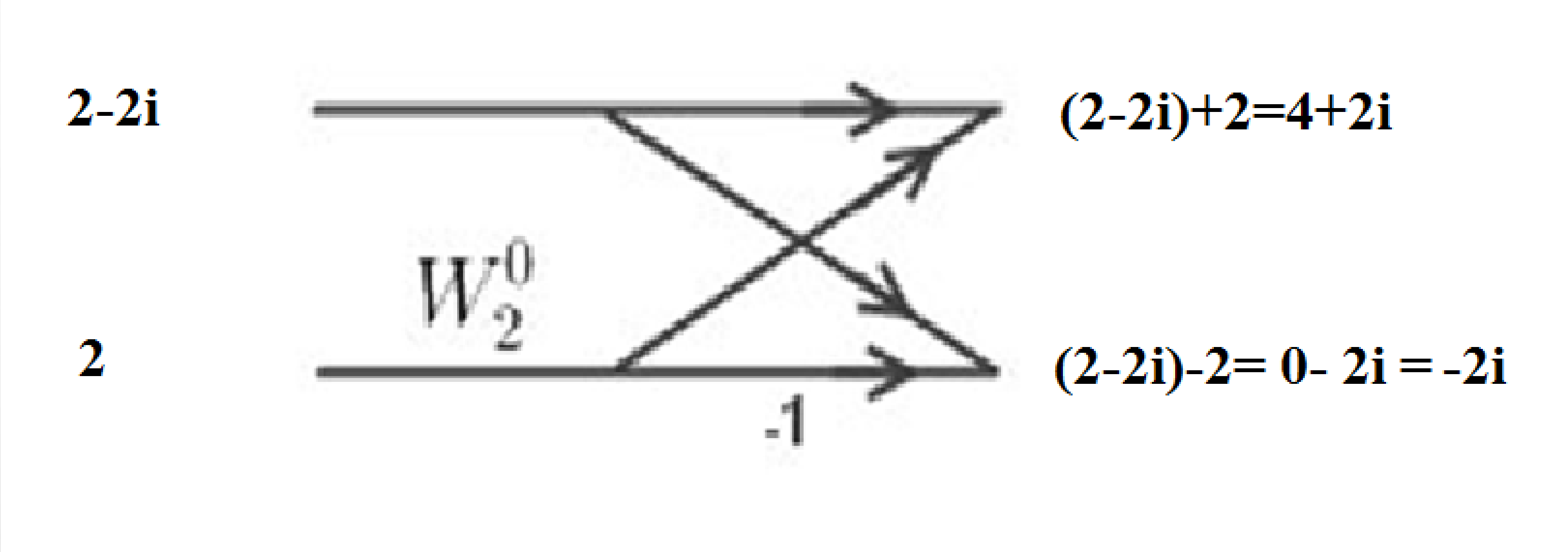
X(n)=[0 1]



EXAMPLE

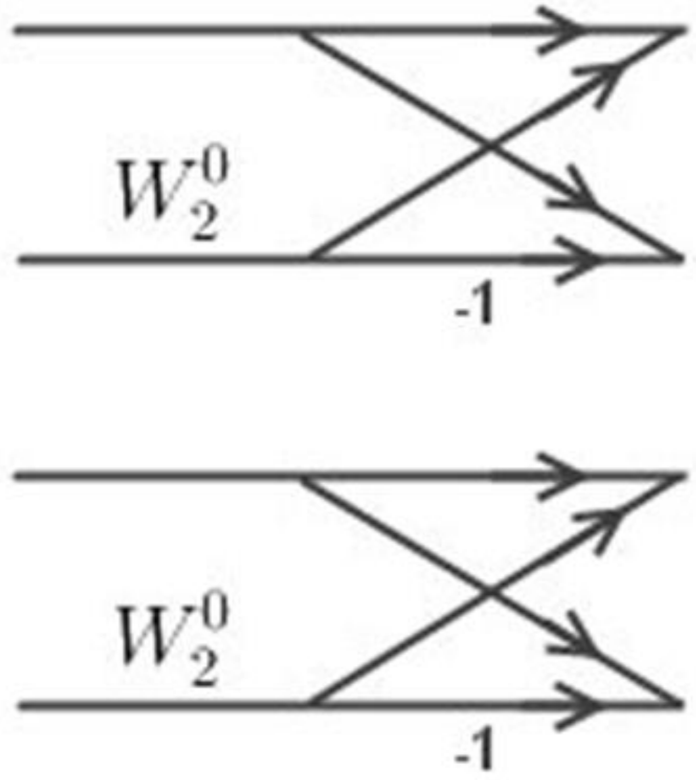
Find the FFT for the signal X(n)

X(n)=[2-2i 2]



The X(k) = [4+2i   -2i]

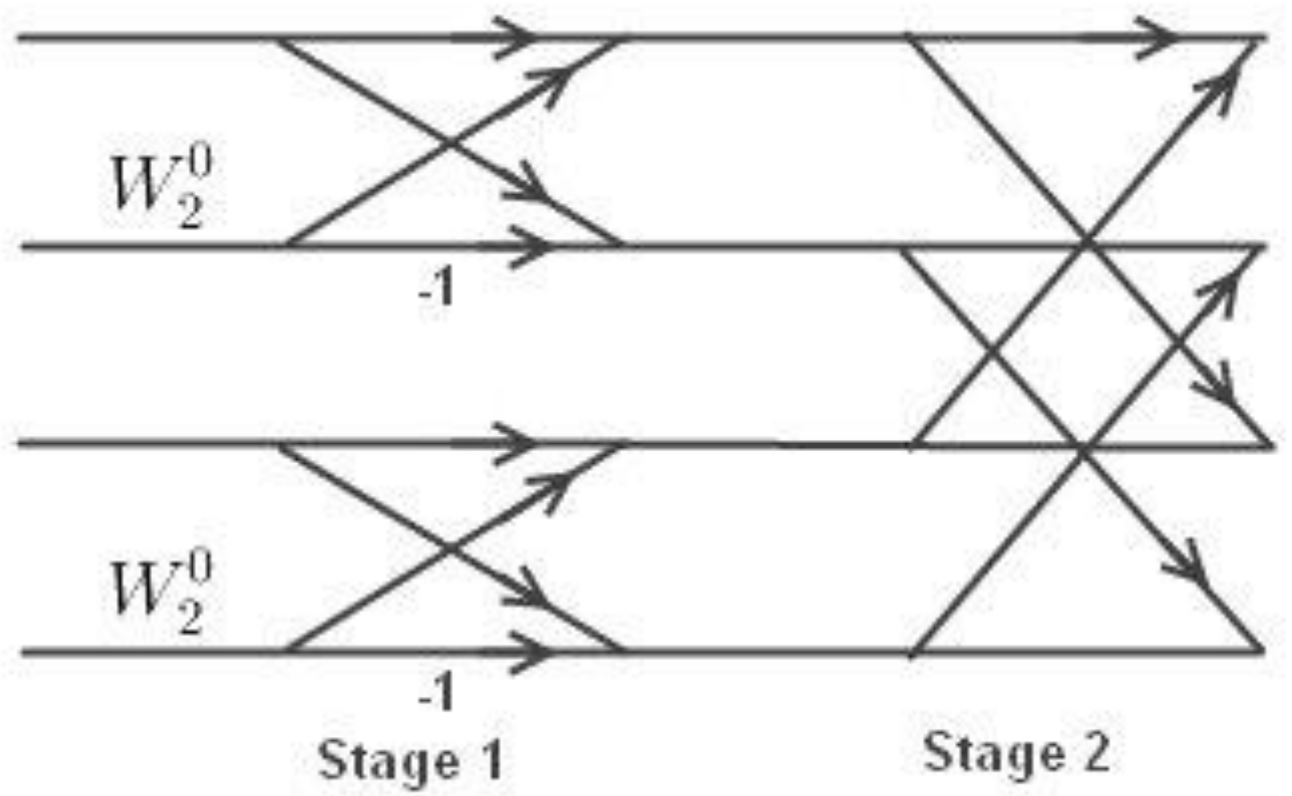
HOW TO CREATE A 4 POINT INPUT BUTTERFLY



Step 1:
Create 2 2-input butterflies.

HOW TO CREATE A 4 POINT INPUT BUTTERFLY

Step 2: Extend out the lines and then connect the bottom butterfly to the top and the top to the bottom.



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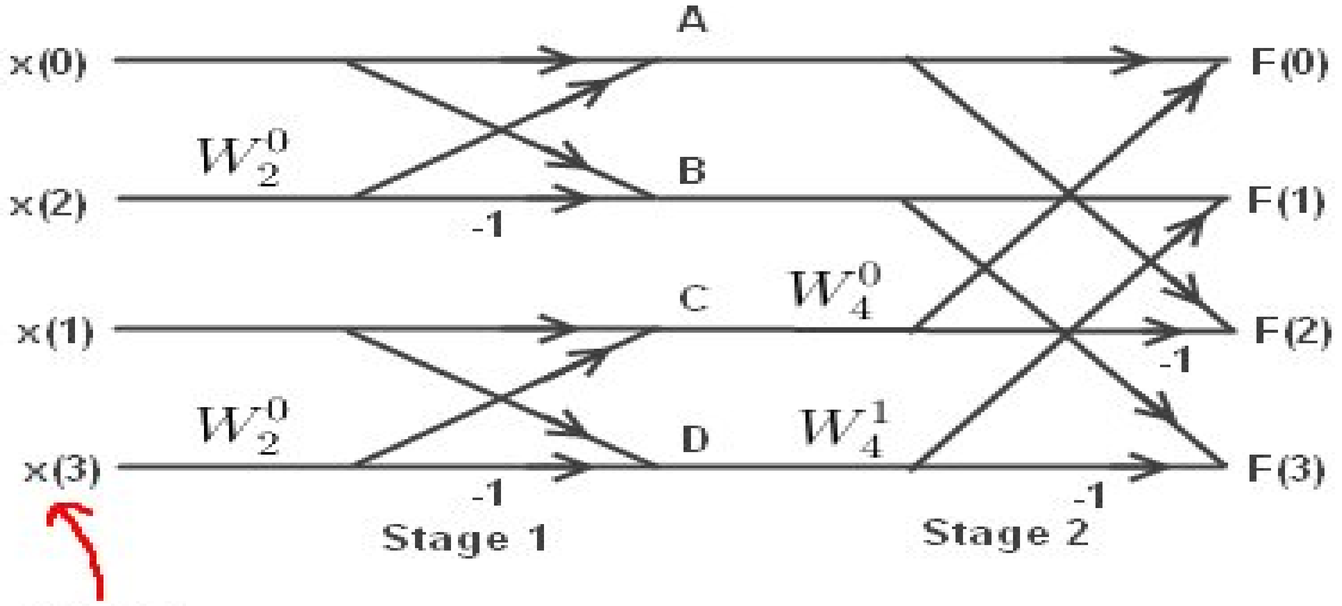
HOW TO CREATE A 4 POINT INPUT BUTTERFLY

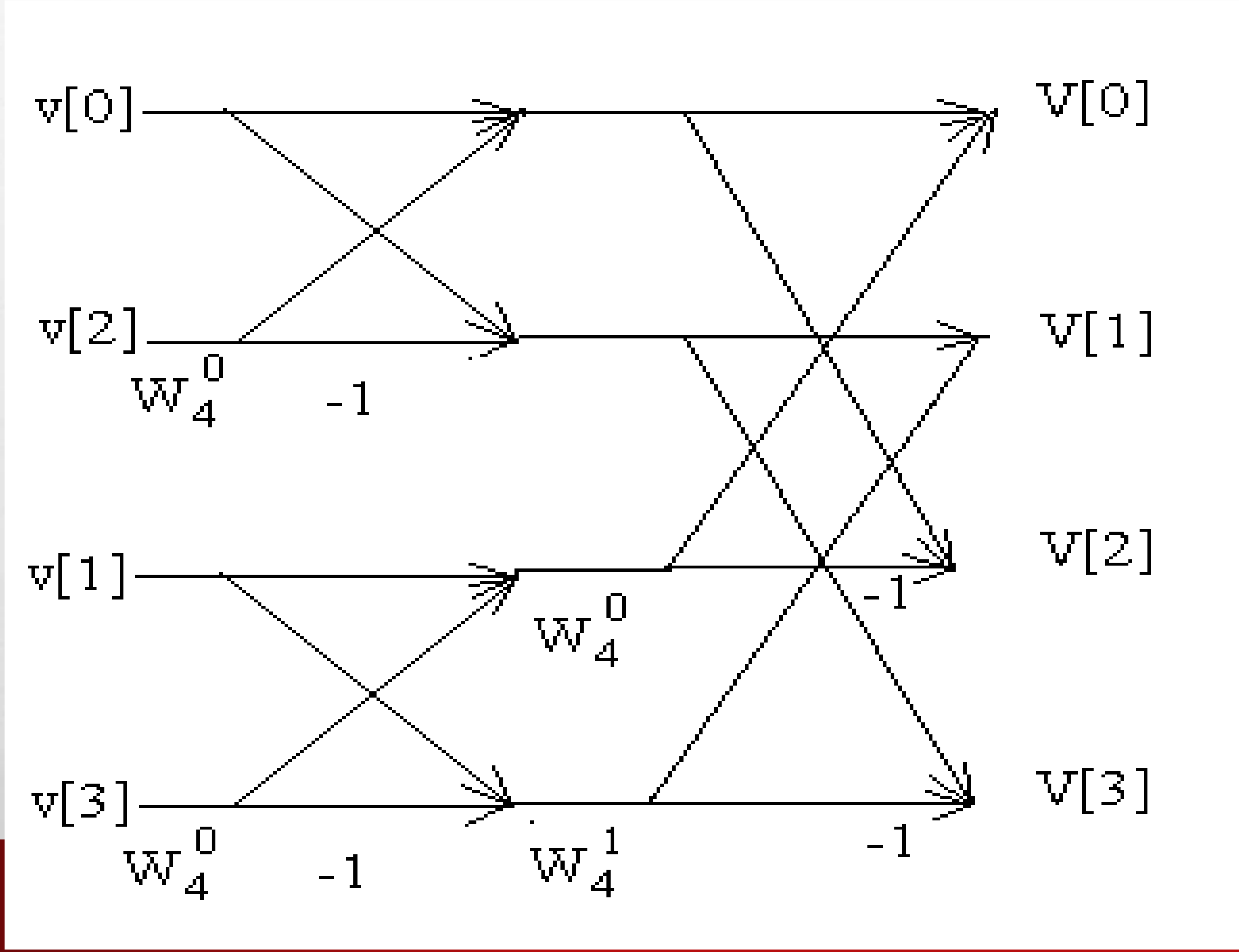
Step 3: Label the input and output values. Label the bottom half of the diagram with W base 4 values, and powers of 0, 1 in order. Note Stage 1 has W base 2, and stage 2 has W base 4. This continues in binary fashion 2, 4, 8, 16 as you add more stages to the butterfly.



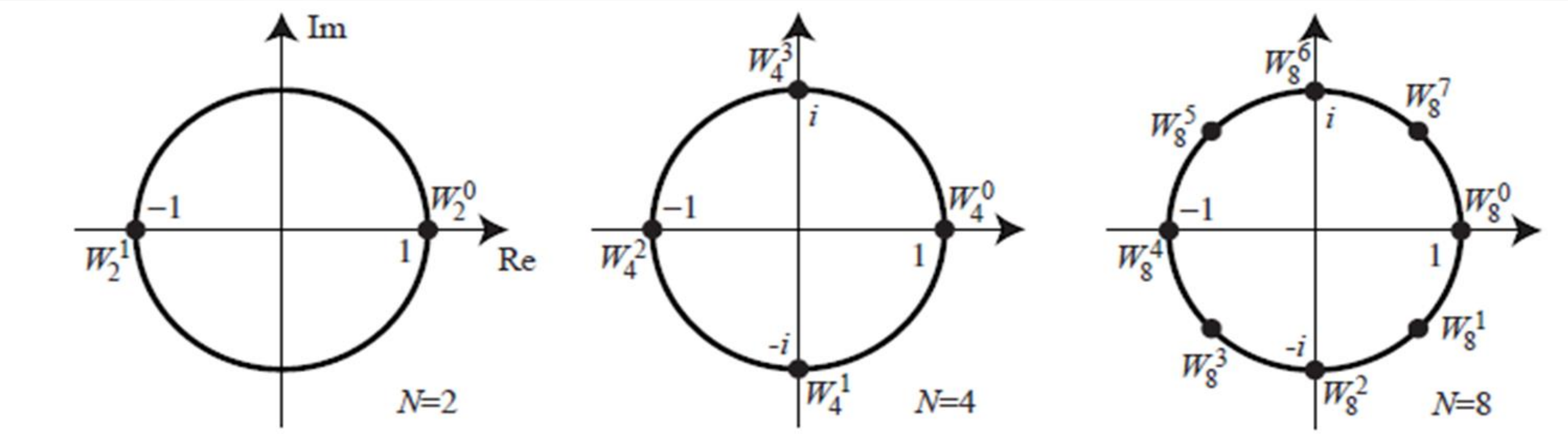
Note the reverse bit ordering of input values.

This is the completed 4 input butterfly.

HOW TO CREATE A 4 POINT INPUT BUTTERFLY



HOW TO CREATE A 4 POINT INPUT BUTTERFLY



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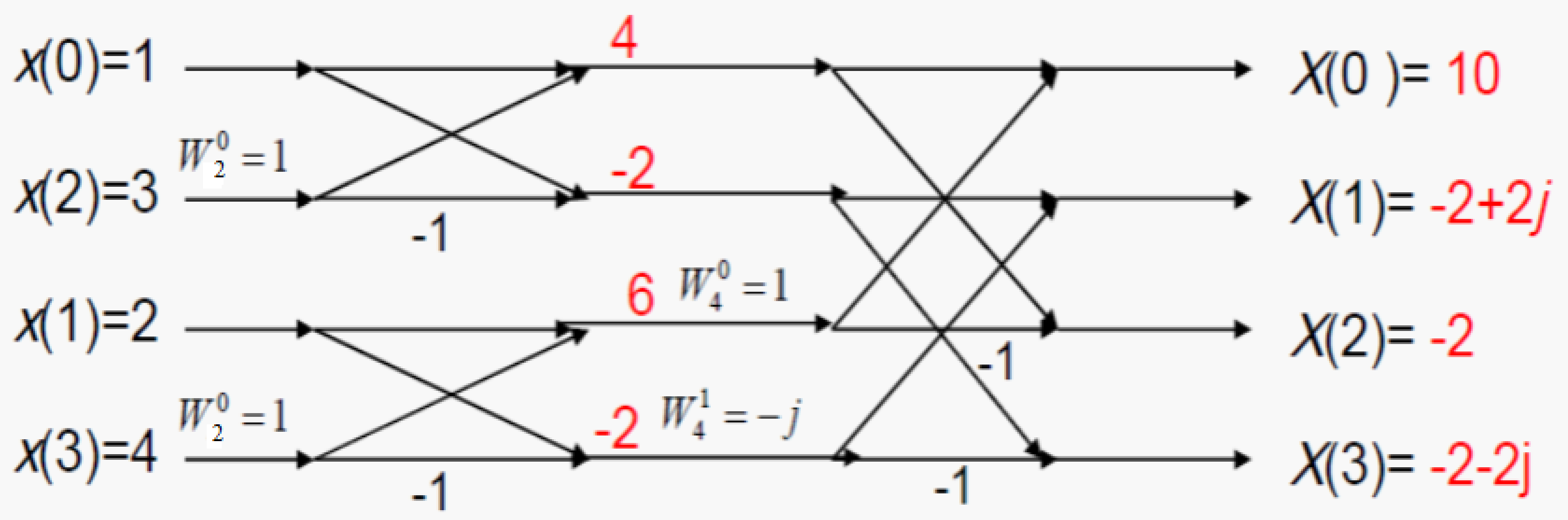
EXAMPLE

Find the FFT for the signal X(n)

X(n)=[1 2 3 4]

X_e(n)=[1 3] **X_{odd}(n)=[2 4]**

Solution



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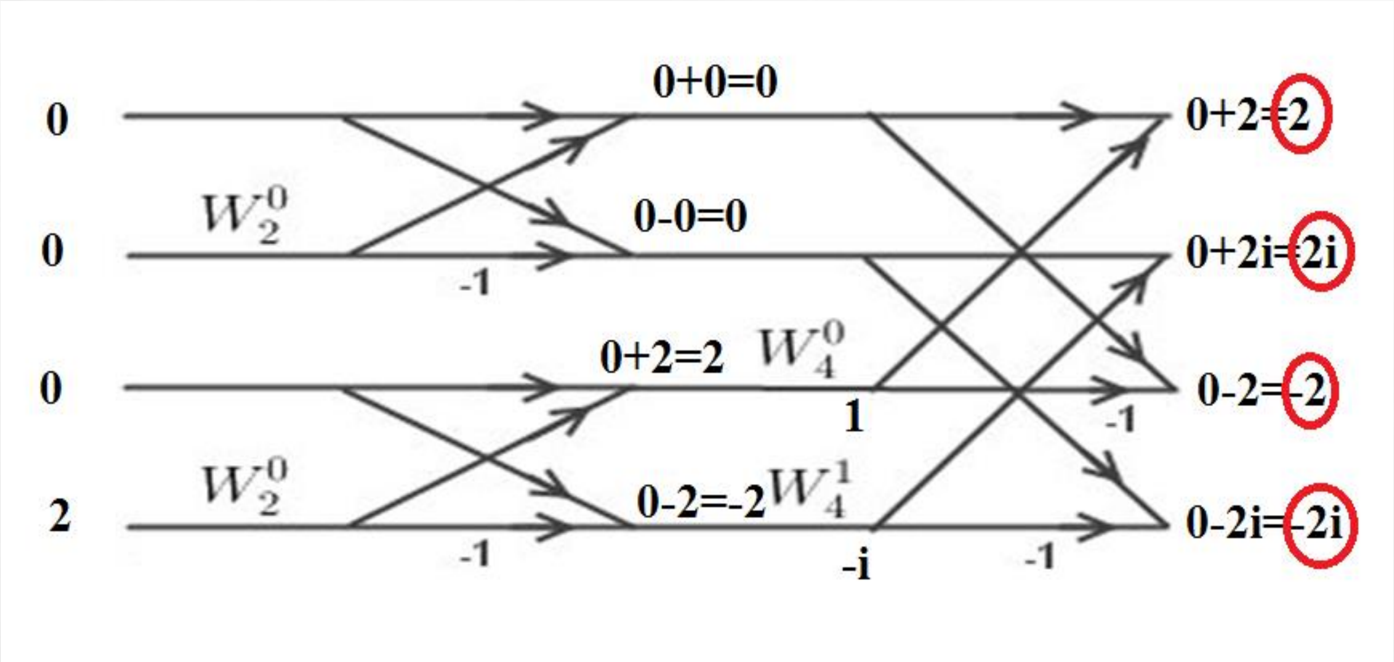
EXAMPLE

Find the FFT for the signal X(n)

X(n)=[0 0 0 2]

$X_e(n)$=[0 0] $X_{odd}(n)$=[0 2]



X(k)=[2 2i -2 -2i]

EXAMPLE

Find the FFT for the signal X(n)

X(n)=[x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)]

N=8

X_e(n)=[x(0), x(2), x(4), x(6)]   N=4

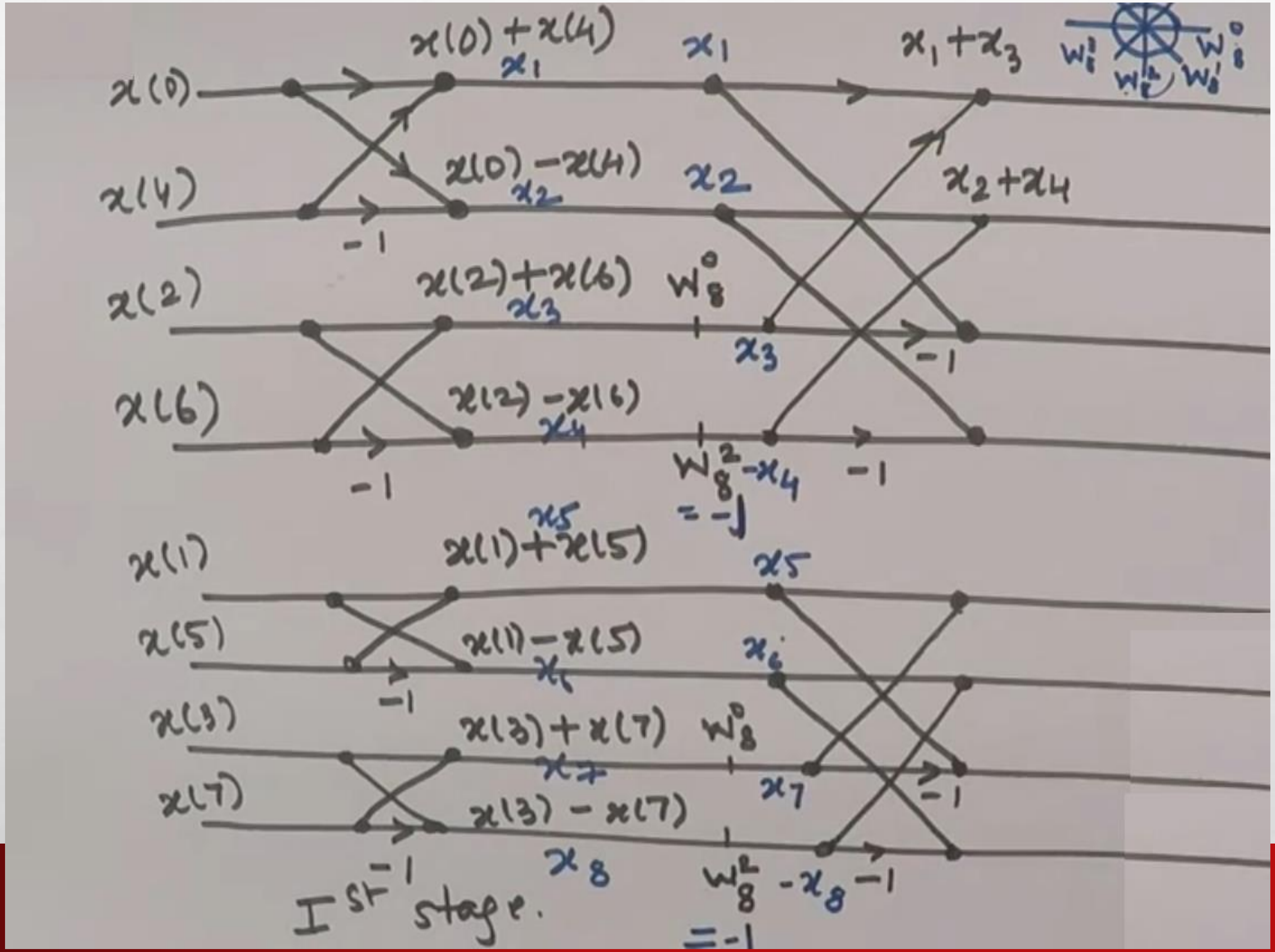
X_o (n)=[x(1), x(3), x(5), x(7)]   N=4

2pt

$x_1(n) = \{x(0), x(4)\}$ $x_2(n) = \{x(2), x(6)\}$

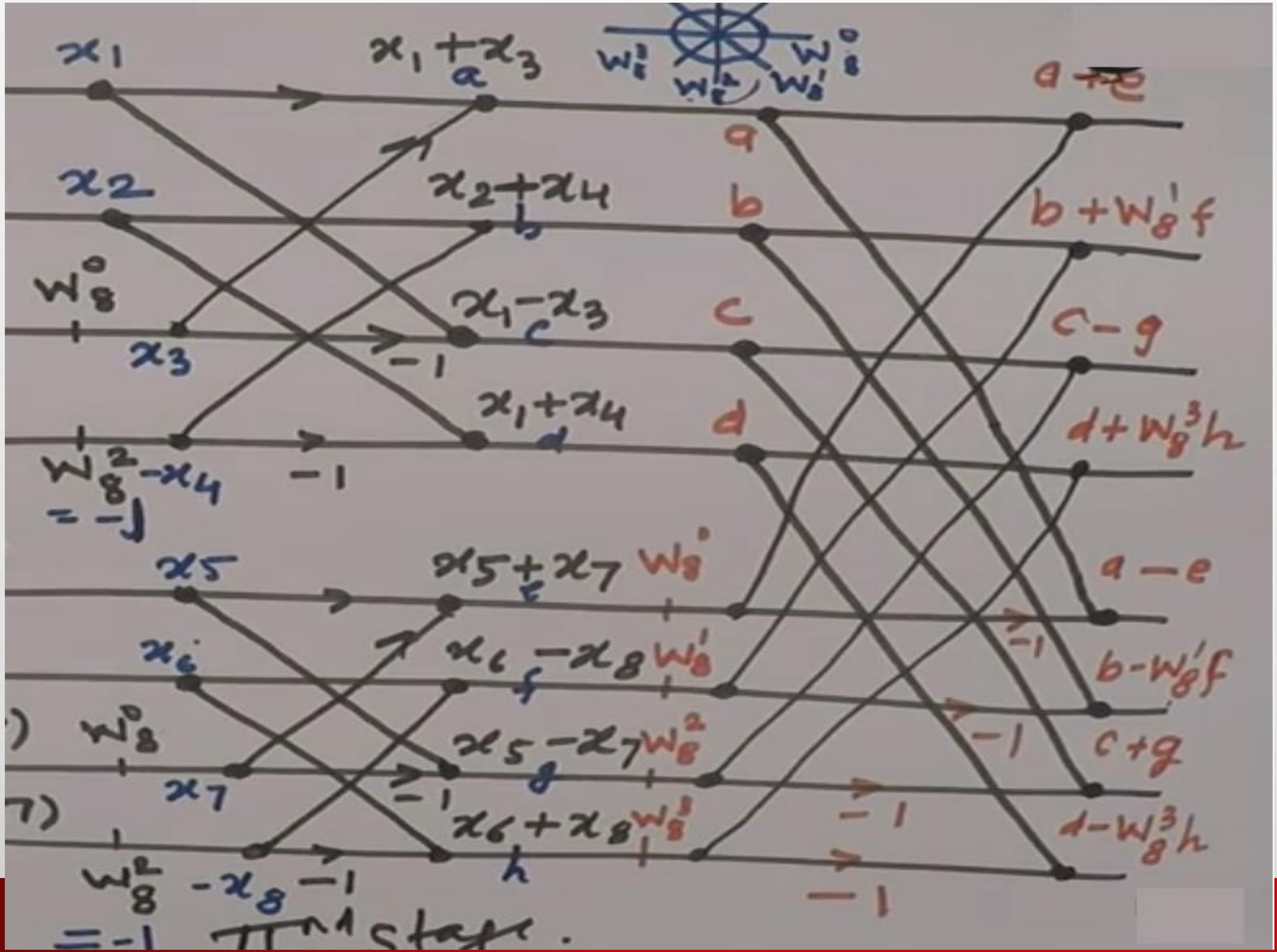
$x_3(n) = \{x(1), x(5)\}$ $x_4(n) = \{x(3), x(7)\}$

EXAMPLE



x(0)
x(4)
x(2)
x(6)
x(1)
x(5)
x(3)
x(7)
x(0) + x(4)
x1
x(0) − x(4)
x2
x(2) + x(6)
x3
x(2) − x(6)
x4
x(1) + x(5)
x5
x(1) − x(5)
x6
x(3) + x(7)
x7
x(3) − x(7)
x8
-1
W8^0
W8^2 -x4
= -j
x1 + x3
x2 + x4
I st stage.

EXAMPLE



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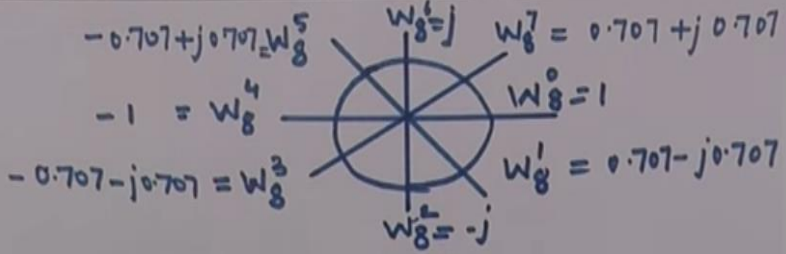
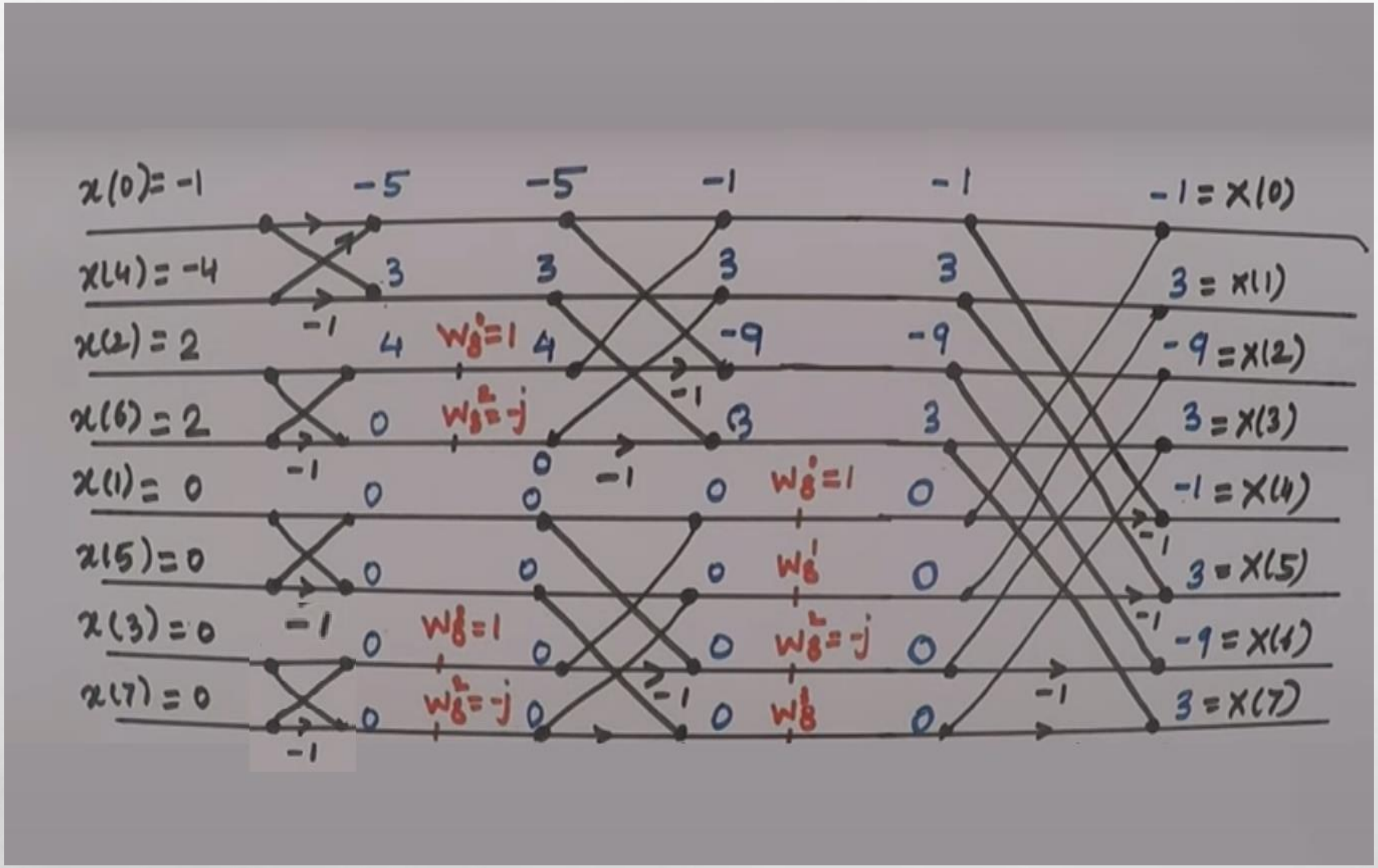
EXAMPLE

$x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$ $\quad X(k) = ?$



EXAMPLE



EXAMPLE

Then the X(k) in frequency domain is

X(k)= [-1 3 -9 3 -1 3 -9 3]

of the signal x(n) in time domain

X(n)=[-1 0 2 0 -4 0 2 0]