AL FURAT AL AWSAT TECHNICAL UNIVERSITY
NAJAF COLLEGE OF TECHNOLOGY
DEPARTMENT OF AVIONICS ENGINEERING

DIGITAL SIGNAL PROCESSING

3rd YEAR

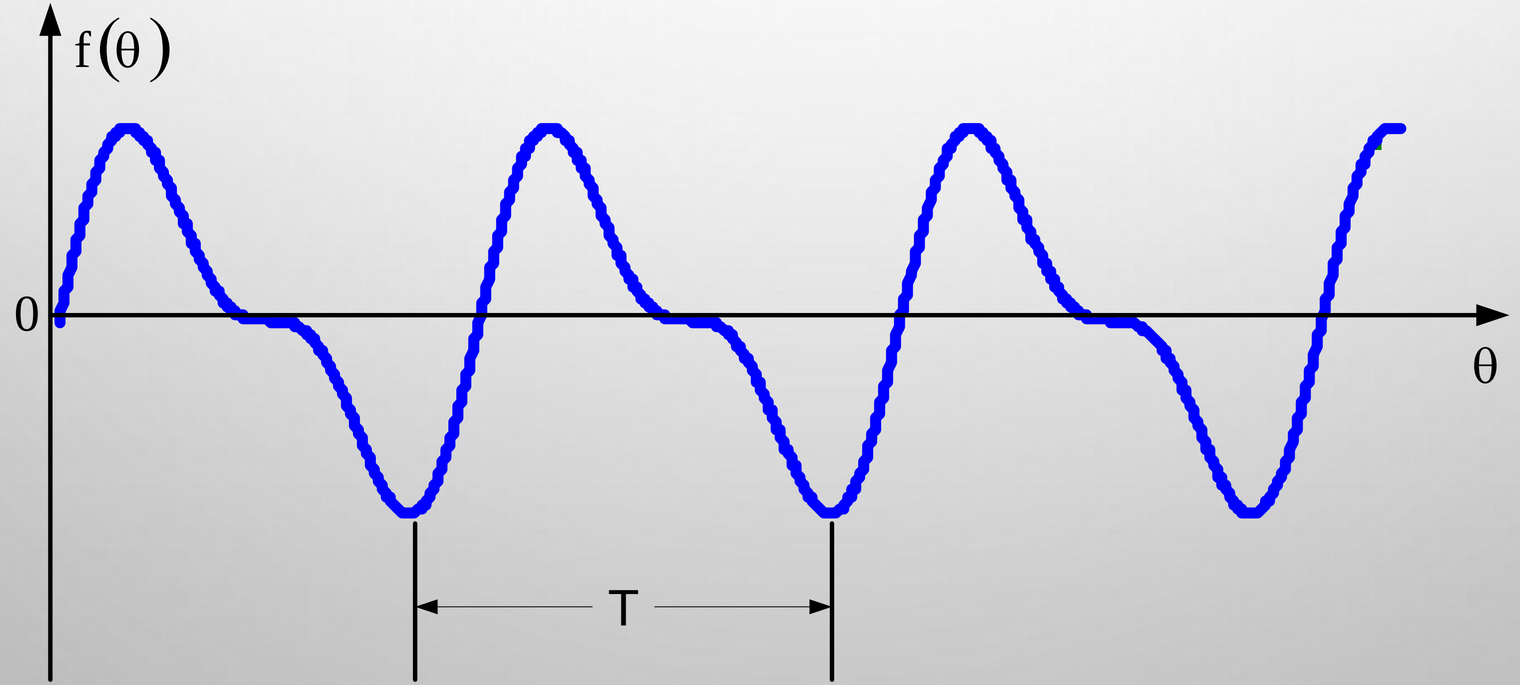
BY

RUAA SHALLAL ANOOZ

PERIODIC FUNCTIONS

A function $f(\theta)$ **is periodic if it is defined for all real** θ **and if there is some positive number,**

T **such that** $f(\theta + T) = f(\theta)$



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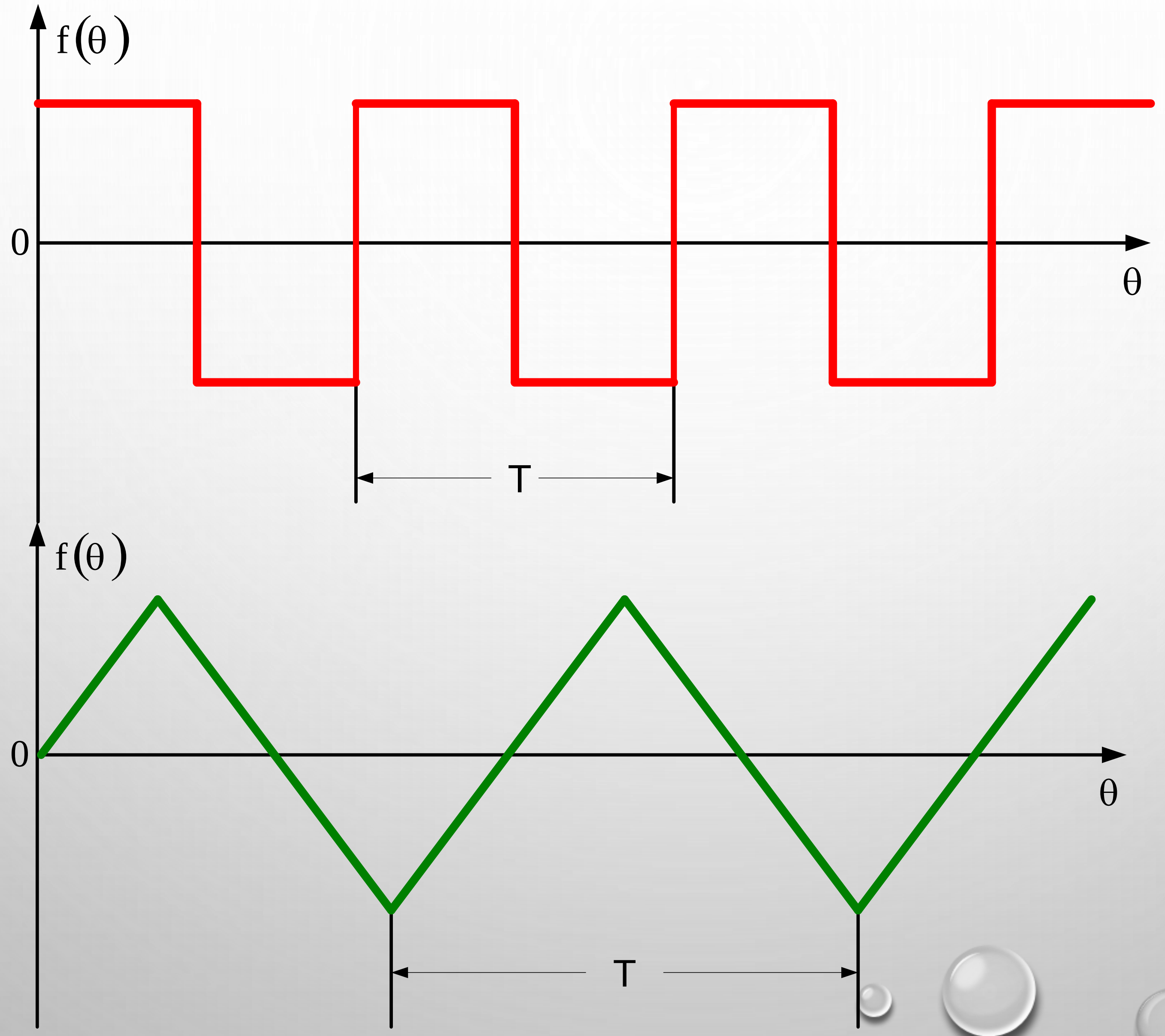
PERIODIC FUNCTIONS



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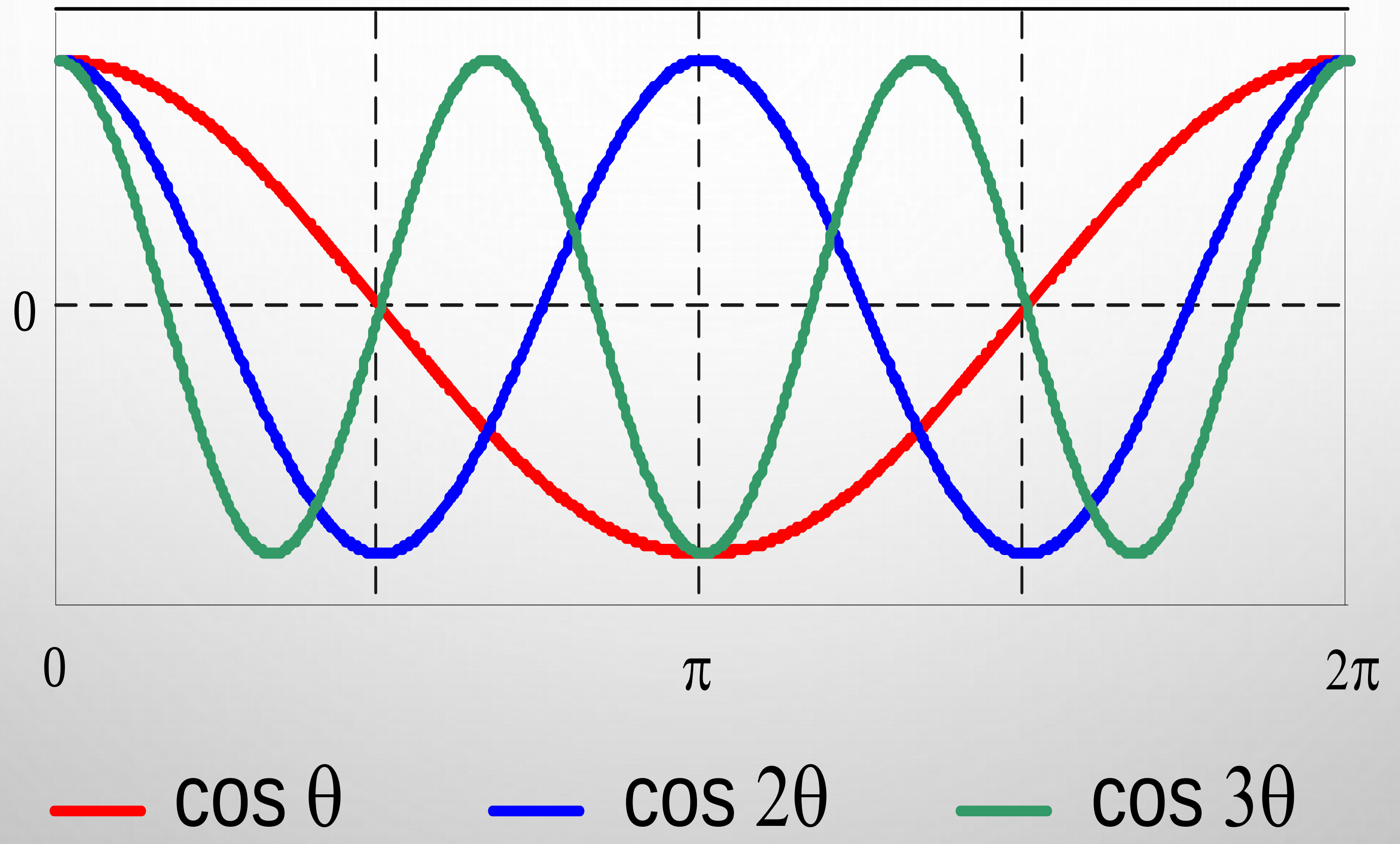

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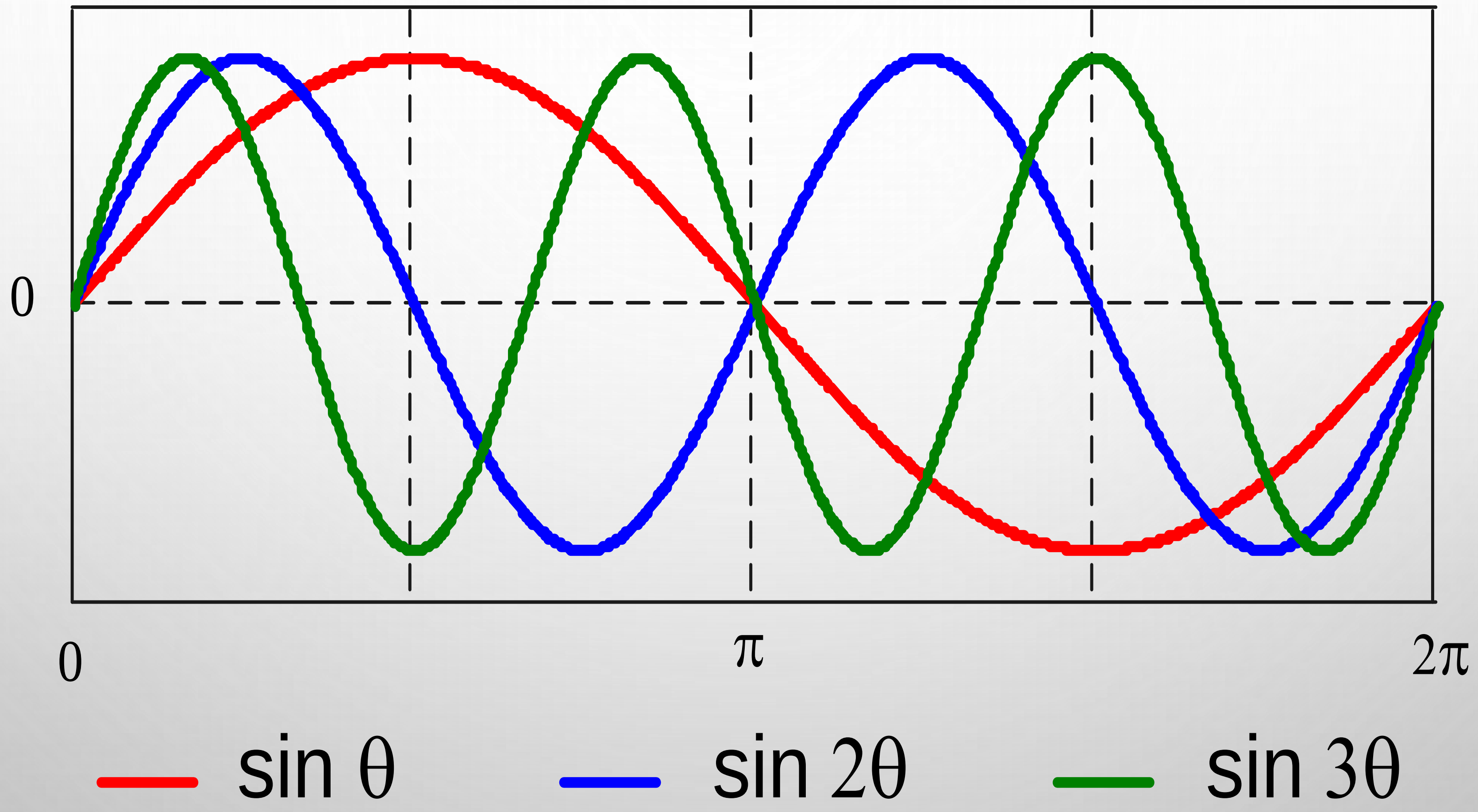
PERIODIC FUNCTIONS



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FOURIER SERIES

$f(\theta)$ **be a periodic function with period** 2π

The function can be represented by a trigonometric series as:

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\theta + \sum_{n=1}^{\infty} b_n \sin n\,\theta \quad ....(1)$$

SOME SPECIAL CASES OF TRIGONOMETRIC INTEGRAL THAT USED IN THE FOURIER SERIES

1. $\int_0^{2\pi} \sin(nt)\, dt = 0$ where n is any integer number

2. $\int_0^{2\pi} \cos(nt)\, dt = 0$ where n is any integer number doesn′t equal zero

3. $\int_0^{2\pi} \sin(nt)\cos(mt)\, dt = 0$ where n, m are any integer numbers

4. $\int_0^{2\pi} \sin(nt)\sin(mt)\, dt = 0$ where m is not equal the integer number n or $-$n

5. $\int_0^{2\pi} (\sin(mt))^2 dt = \pi$ where m $=$ n and it is not equal zero

6. $\int_0^{2\pi} \cos(nt)\cos(mt)\, dt = 0$ where m is not equal the integer number n or $-$n

7. $\int_0^{2\pi} (cos(mt))^2 dt = \pi$ where m $=$ n and it is not equal zero

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FOURIER SERIES

We want to determine the coefficients, a_0, a_n and b_n

1. **Determine a_0**

Integrate both sides of (1) from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\theta + \sum_{n=1}^{\infty} b_n \sin n\,\theta \right] d\theta$$

$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = \int_{-\pi}^{\pi} a_0\, d\theta + \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} a_n \cos n\,\theta \right) d\theta \quad + \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} b_n \sin n\,\theta \right) d\theta$$

$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = \int_{-\pi}^{\pi} a_0\, d\theta + 0 + 0$$

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$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = 2\pi a_0 + 0 + 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

a_0 is the average (dc) value of the function, $f(\theta)$

You may integrate both sides of (1) from 0 to 2π instead.

$$\int_{0}^{2\pi} f(\theta)\, d\theta = \int_{0}^{2\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\theta + \sum_{n=1}^{\infty} b_n \sin n\,\theta\right] d\theta$$

It is alright as long as the integration is performed over one period.

FOURIER SERIES

$$\int_0^{2\pi} f(\theta)\,d\theta = \int_0^{2\pi} a_0\,d\theta + \int_0^{2\pi}\left(\sum_{n=1}^{\infty} a_n \cos n\,\theta\right)d\theta \quad + \int_0^{2\pi}\left(\sum_{n=1}^{\infty} b_n \sin n\,\theta\right)d\theta$$

$$\int_0^{2\pi} f(\theta)\,d\theta = \int_0^{2\pi} a_0\,d\theta + 0 + 0$$

$$\int_0^{2\pi} f(\theta)\,d\theta = 2\pi a_0 + 0 + 0$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

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2. Determine a_n

Multiply (1) by $\cos m\theta$ and then Integrate both sides from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(\theta)\cos m\theta d\theta = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta\right]\cos m\theta\, d\theta$$

Let us do the integration on the right-hand-side one term at a time.

First term, $\int_{-\pi}^{\pi} a_0 \cos m\theta d\theta = 0$

Second term, $\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n \cos n\theta \cos m\theta\, d\theta$

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Therefore, $\int_{-\pi}^{\pi} f(\theta)\cos m\,\theta d\theta = a_n\pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos m\,\theta d\theta \quad n = m$$

3. Determine b_n

Multiply (1) by $\sin m\,\theta$ and then Integrate both sides from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(\theta)\sin m\,\theta d\theta = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\theta + \sum_{n=1}^{\infty} b_n \sin n\,\theta\right]\sin m\,\theta\, d\theta$$

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Let us do the integration on the right-hand-side one term at a time.

First term, $\int_{-\pi}^{\pi} a_0 \sin m\,\theta d\theta = 0$

Second term, $\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos n\,\theta \sin m\,\theta\, d\theta = 0$

Third term, $\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \sin n\,\theta \sin m\,\theta\, d\theta = b_m \pi$

Therefore,

$$\int_{-\pi}^{\pi} f(\theta) \sin m\,\theta d\theta = b_m \pi$$

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin m\,\theta d\theta \quad m = n$$

FOURIER SERIES

The coefficients are:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos m\,\theta d\theta \quad m = 1, 2, \cdots$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \sin m\,\theta d\theta \quad m = 1, 2, \cdots$$

We can write *n* in place of *m*:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos n\,\theta d\theta \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \sin n\,\theta d\theta \quad n = 1, 2, \cdots$$

FOURIER SERIES

The integrations can be performed from $\mathbf{0}$ to $\mathbf{2\pi}$ instead.

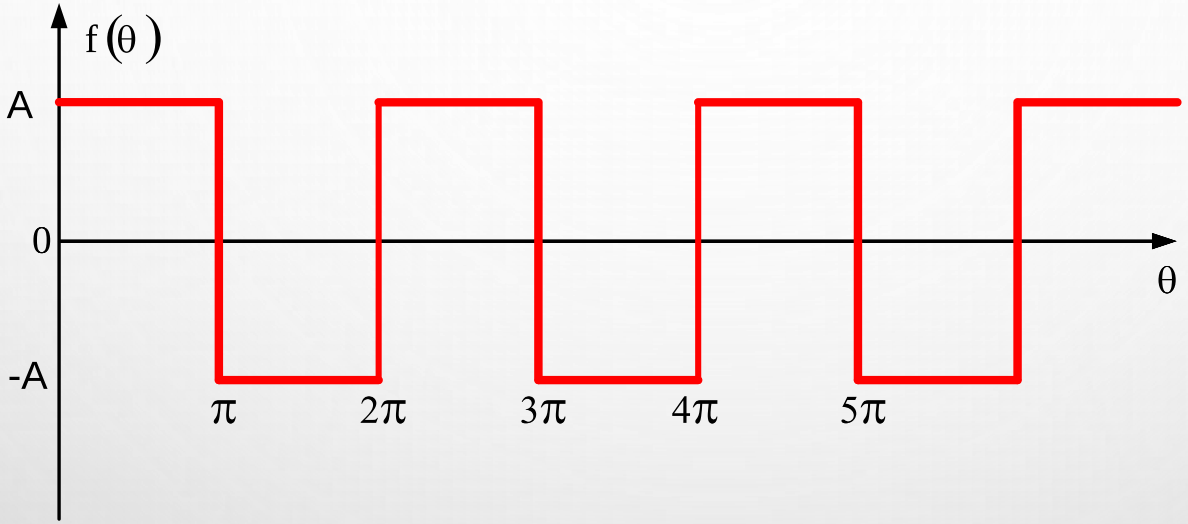
$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\,\theta d\theta \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\,\theta d\theta \quad n = 1, 2, \cdots$$

Example

Find the Fourier series of the following periodic function.



$$f(\theta) = A \quad when \quad 0 < \theta < \pi$$
$$= -A \quad when \quad \pi < \theta < 2\pi$$

$$f(\theta + 2\pi) = f(\theta)$$

Example

Sol:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} f(\theta)\,d\theta + \int_{\pi}^{2\pi} f(\theta)\,d\theta\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} A\,d\theta + \int_{\pi}^{2\pi} -A\,d\theta\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\,\theta d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A\cos n\,\theta d\theta + \int_{\pi}^{2\pi} (-A)\cos n\,\theta d\theta\right]$$

$$= \frac{1}{\pi}\left[A\frac{\sin n\,\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[-A\frac{\sin n\,\theta}{n}\right]_{\pi}^{2\pi} = 0$$

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Example

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\,\theta d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A\sin n\,\theta d\theta + \int_{\pi}^{2\pi} (-A)\sin n\,\theta d\theta\right]$$

$$= \frac{1}{\pi}\left[-A\frac{\cos n\,\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[A\frac{\cos n\,\theta}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{A}{n\pi}[-\cos n\,\pi + \cos 0 + \cos 2\,n\pi - \cos n\,\pi]$$

$$b_n = \frac{A}{n\pi}[-\cos n\,\pi + \cos 0 + \cos 2\,n\pi - \cos n\,\pi]$$

$$= \frac{A}{n\pi}[1 + 1 + 1 + 1]$$

$$= \frac{4A}{n\pi} \quad \text{when n is odd}$$

$$b_n = \frac{A}{n\pi}[-\cos n\,\pi + \cos 0 + \cos 2\,n\pi - \cos n\,\pi]$$

$$= \frac{A}{n\pi}[-1 + 1 + 1 - 1]$$

$$= 0 \quad \text{when n is even}$$

Example

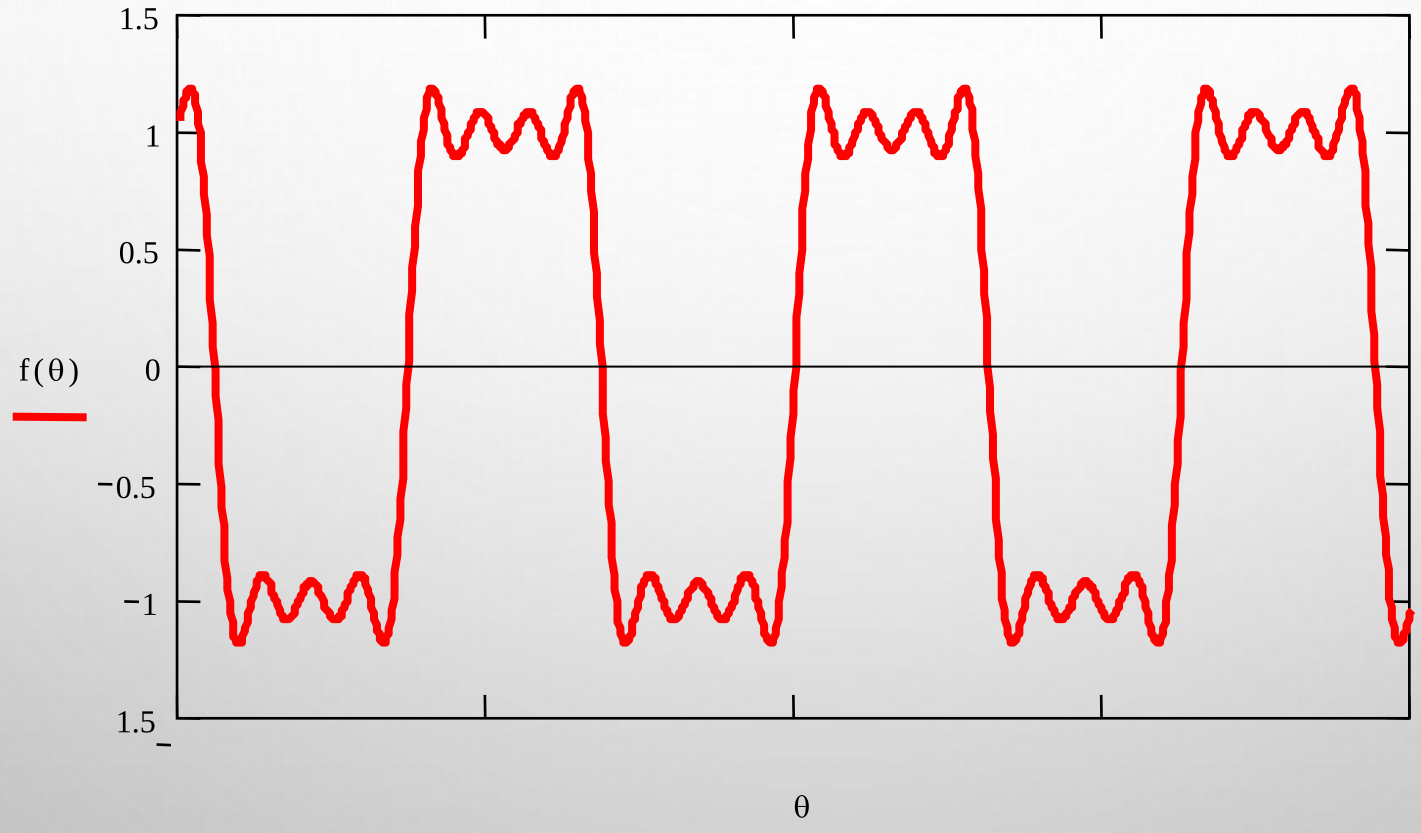
Therefore, the corresponding Fourier series is

$$\frac{4A}{\pi}\left(\sin\theta + \frac{1}{3}\sin 3\,\theta + \frac{1}{5}\sin 5\,\theta + \frac{1}{7}\sin 7\,\theta + \cdots\right)$$

In writing the Fourier series we may not be able to consider infinite number of terms for practical reasons. The question therefore, is – how many terms to consider?

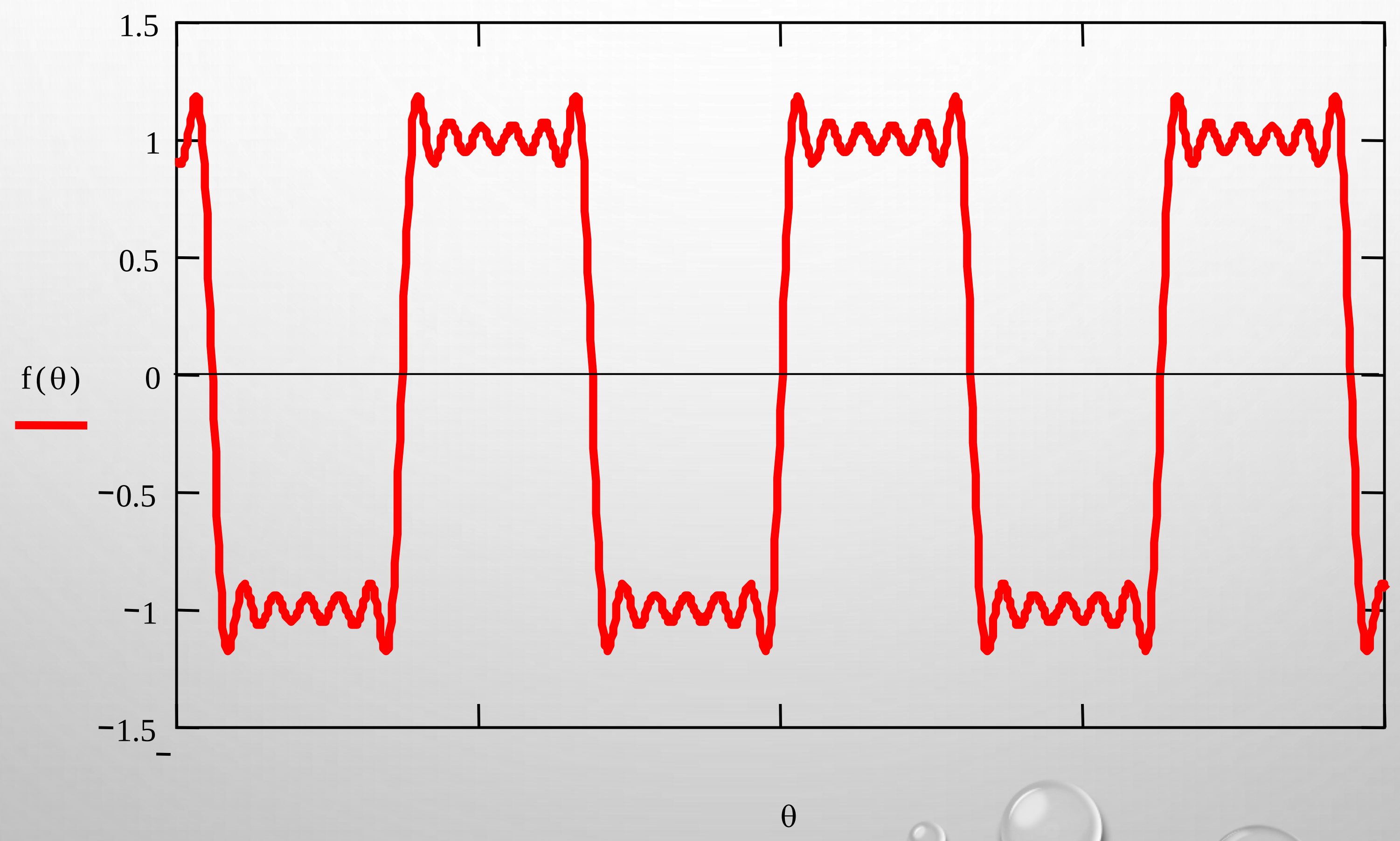
Example

When we consider 4 terms as shown in the previous slide, the function looks like the following.



Example

When we consider 6 terms, the function looks like the following.



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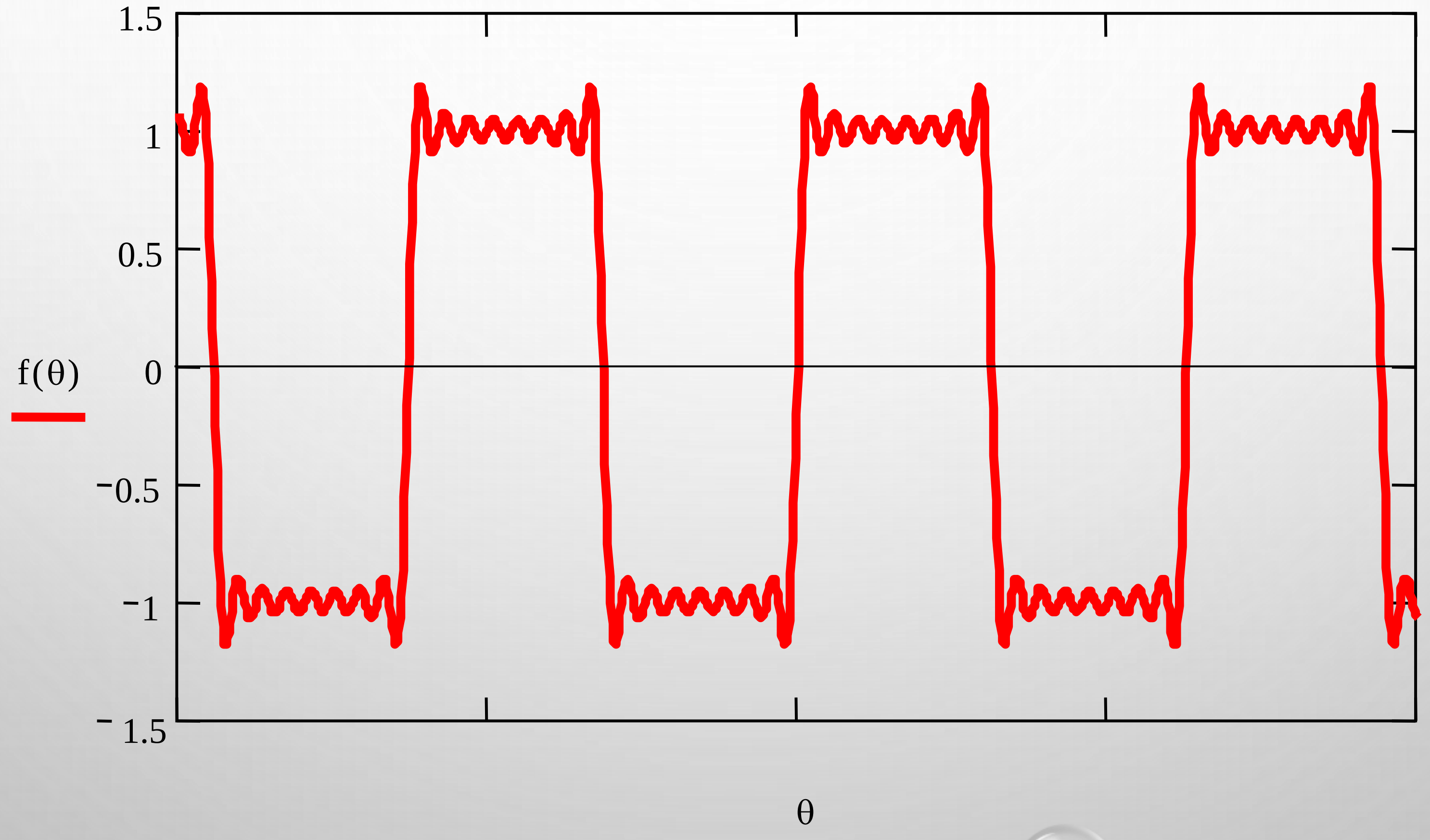
Example

When we consider 8 terms, the function looks like the following.



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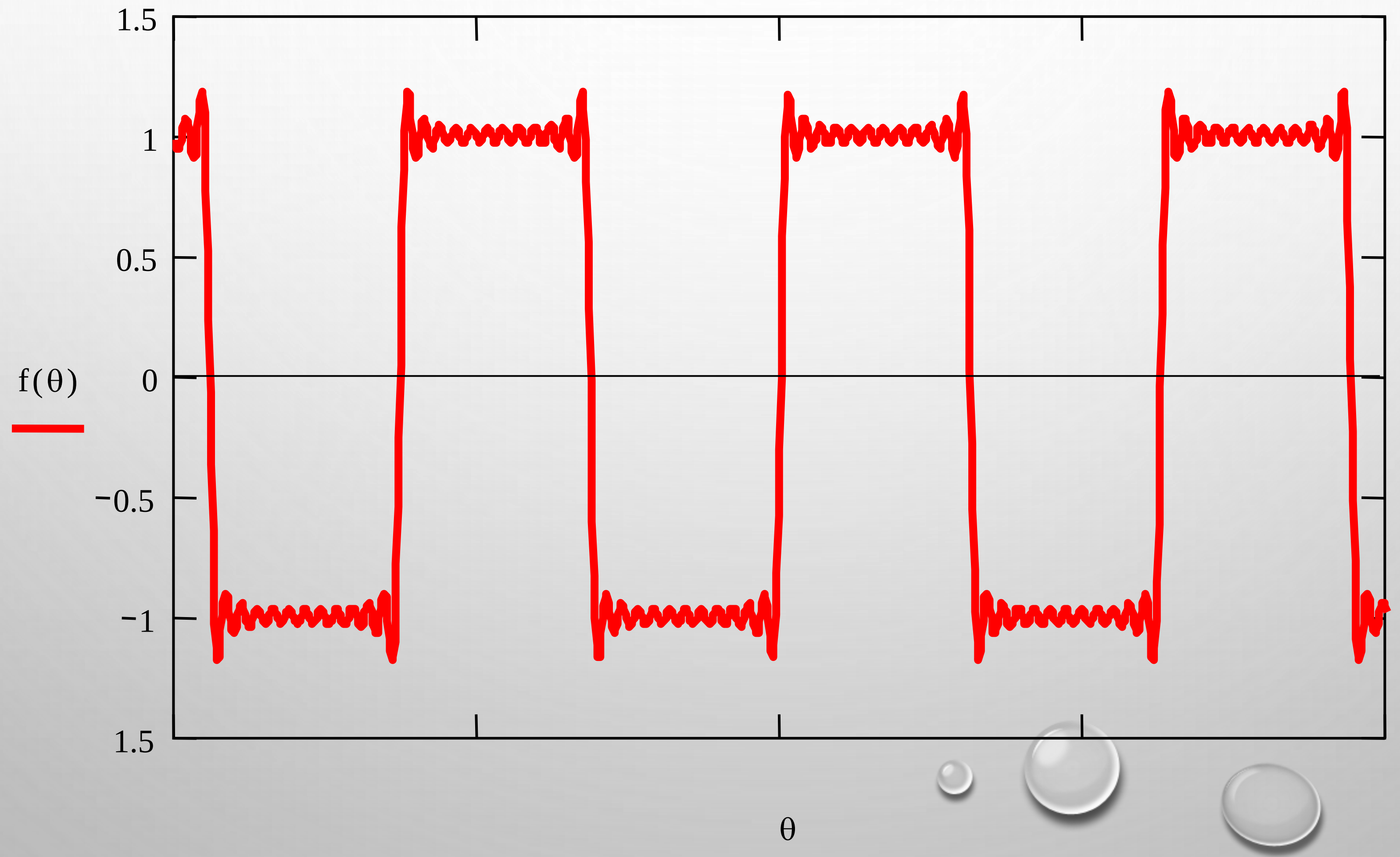
Example

When we consider 12 terms, the function looks like the following.



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