PROBABILITY, SIGNALS & SYSTEMS

BY: RUAA SHALLAL ANOOZ

LAPLACE TRANSFORM

$$F(s) = L\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st}dt$$

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

- t is real, s is complex!
- Inverse requires complex analysis to solve.
- Note “transform”: f(t) ⟶ F(s), where t is integrated and s is variable.
- Conversely F(s) ⟶ f(t), t is variable and s is integrated.

APPLICATIONS OF LAPLACE TRANSFORM

- Easier than solving differential equations
  - Used to describe system behavior
  - We assume LTI systems
  - Uses S-domain instead of frequency domain
- Applications of Laplace Transforms/
  - Circuit analysis
    - Easier than solving differential equations
    - Provides the general solution to any arbitrary wave (not just LRC)
      - Transient
      - Sinusoidal steady-state-response (Phasors)
  - Signal processing
  - Communications

EXAMPLES

Ex1: let $f(t) = 1$ find F(S)?

Sol:

$$F(s) = \int_0^\infty e^{-st} dt = -\frac{1}{s}(0-1) = \frac{1}{s}$$

Ex2: let $f(t) = e^{-at}$ find F(S)?

Sol:

$$F(s) = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt = \frac{1}{s+a}$$

EXAMPLES

Ex3: let $f(t) = \sin t$ find F(S)?

Sol:

$$F(s) = \int_0^\infty e^{-st}\sin(t)dt \longleftarrow$$ **Integrate by parts**

REMEMBER $\int udv = uv - \int vdu$

let $\boldsymbol{u = e^{-st}, du = -se^{-st}dt}$
$\boldsymbol{dv = \sin(t)dt, v = -\cos(t)}$

$$\therefore \int_0^\infty e^{-st}\sin(t)dt = [-e^{-st}\cos(t)\Big]_0^\infty - s\int_0^\infty e^{-st}\cos(t)dt =$$

$$-e^{-st}(1) - s\int_0^\infty e^{-st}\cos(t)dt$$

let $\boldsymbol{u = e^{-st}, du = -se^{-st}dt}$
$\boldsymbol{dv = \cos(t)dt, v = \sin(t)}$

$$\therefore \int_0^\infty e^{-st}\cos(t)dt =$$

$$[-e^{-st}\sin(t)\Big]_0^\infty + s\int_0^\infty e^{-st}\sin(t)dt = -e^{-st}(0) + s\int_0^\infty e^{-st}\sin(t)dt$$

Substituting, we get:

$$\int_0^\infty se^{-st}\sin(t)dt = 1 + s^2\int_0^\infty e^{-st}\sin(t)dt =$$

$$(1+s^2)\int_0^\infty e^{-st}\sin(t)dt = 1$$

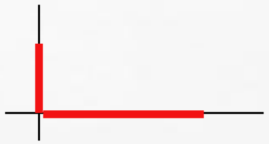
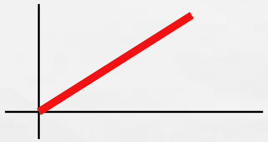
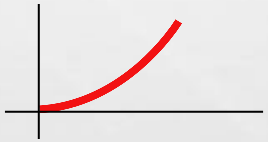
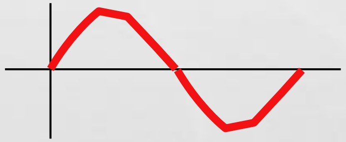
$$\int_0^\infty e^{-st}\sin(t)dt = \frac{1}{1+s^2}$$

LAPLACE TRANSFORM PROPERTIES

Addition/Scaling	$L[af_1(t) \pm bf_2(t)] = aF_1(s) \pm bF_2(s)$
Differentiation	$L\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
Integration	$L\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)dt\right]_{t=0\pm}$
Convolution	$\int_0^t f_1(t-\tau)f_2(\tau)d\tau = F_1(s)F_2(s)$
Initial − value theorem	$f(0+) = \lim_{s\to\infty} sF(s)$
Final − value theorem	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Scaling in Time	$L\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$
Time Shift	$L\{f(t-t_0)u(t-t_0)\} = e^{-st_0}F(s)$
S-plane (frequency) shift	$L\{e^{-at}f(t)\} = F(s+a)$
Multiplication by t^n	$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$

TABLE OF SELECTED LAPLACE TRANSFORMS

Name	$f(t)$		$F(s)$
Impulse	$f(t) = \begin{cases} 1 & t = 0 \\ 0 & t > 0 \end{cases}$		1
Step	$f(t) = 1$		$\frac{1}{s}$
Ramp	$f(t) = t$		$\frac{1}{s^2}$
Exponential	$f(t) = e^{at}$		$\frac{1}{s-a}$
Sine	$f(t) = \sin(\omega t)$		$\frac{1}{\omega^2 + s^2}$

Ruaa Shallal Abbas

MORE TRANSFORMS

$$f(t) = t^n u(t) \Leftrightarrow F(s) = \frac{n!}{s^{n+1}}$$

$$n = 0, f(t) = u(t) \Leftrightarrow F(s) = \frac{0!}{s^1} = \frac{1}{s}$$

$$n = 1, f(t) = tu(t) \Leftrightarrow F(s) = \frac{1!}{s^2}$$

$$n = 5, f(t) = t^5 u(t) \Leftrightarrow F(s) = \frac{5!}{s^6} = \frac{120}{s^6}$$

$$f(t) = \delta(t) \Leftrightarrow F(s) = 1$$

EXAMPLES

Ex: Find $L\{\frac{1}{2}e^{t}-\frac{1}{2}e^{-t}\}$

Sol:

$$=\frac{1}{2}L\{e^{t}\}-\frac{1}{2}L\{e^{-t}\}$$

$$=\frac{1}{2}(\frac{1}{s-1}-\frac{1}{s+1})$$

$$=\frac{1}{2}\left(\frac{(s+1)-(s-1)}{s^{2}-1}\right)$$

$$=\frac{1}{s^{2}-1}$$

EXAMPLES

Ex: Find L{d/dt cos(t)}

Sol:

$$= \frac{s^2}{s^2+1} - f(0^+)$$

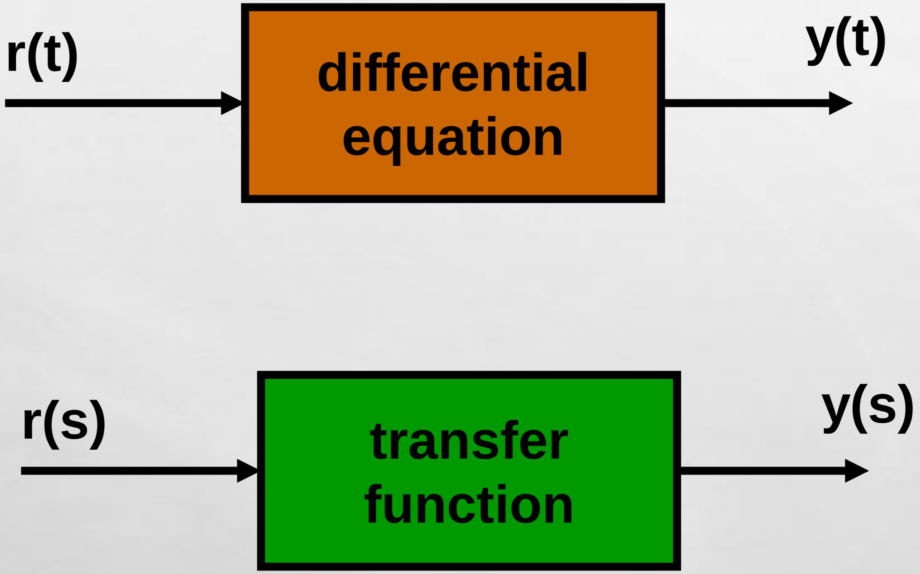
$$= \frac{s^2}{s^2+1} - 1$$

$$= \frac{s^2 - (s^2+1)}{s^2+1}$$

$$\frac{-1}{s^2+1} = L\{-\sin(t)\}$$

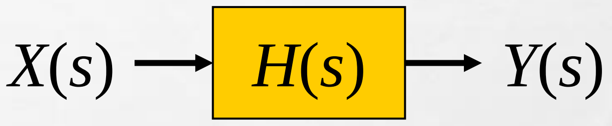
TRANSFER FUNCTIONS

- Definition : a transfer function is an expression that relates the output to the input in the s-domain



TRANSFER FUNCTIONS

- Definition
  - H(s) = Y(s) / X(s)



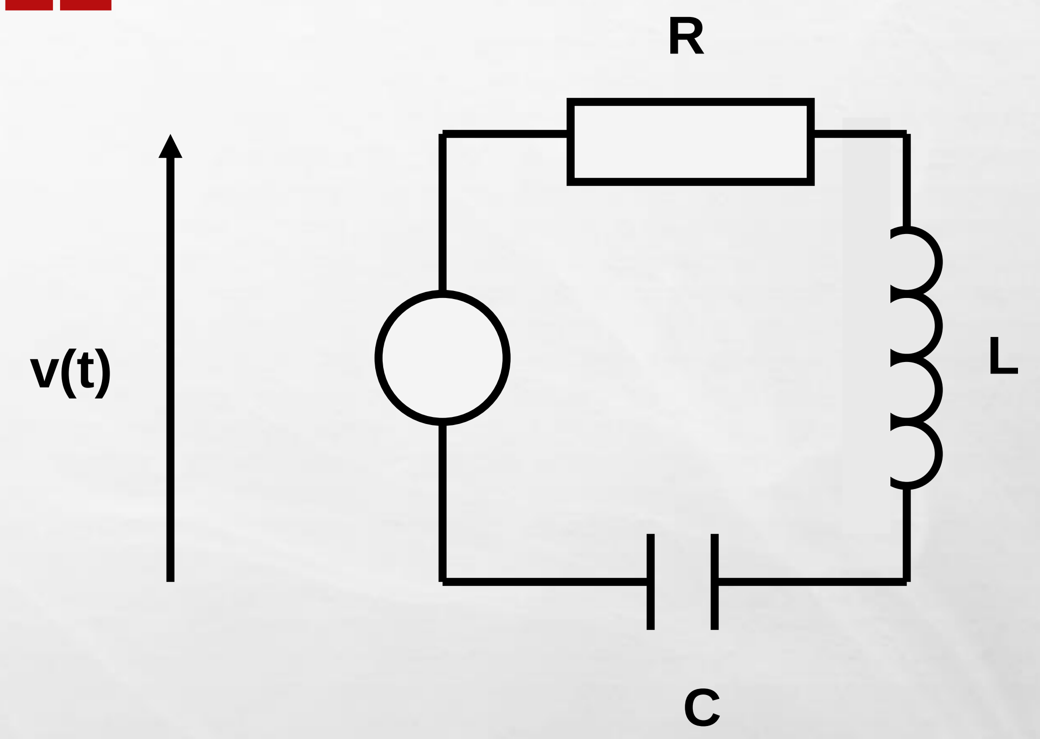
- Relates the output of a linear system (or component) to its input
- Describes how a linear system responds to an impulse
- All linear operations allowed
  - Scaling, addition, multiplication

EXAMPLE

$v(t) = R\,I(t) + 1/C \int I(t)\,dt + L\,di(t)/dt$

$V(s) = [R\,I(s) + 1/(C\,s)\,I(s) + s\,L\,I(s)]$

Note: Ignore initial conditions



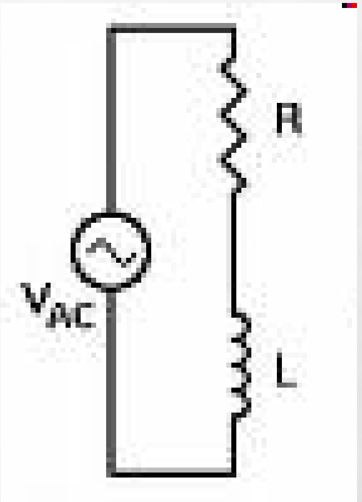
EXAMPLE

- $V(s) = (R + 1/(C s) + s L)\, I(s)$

  $= (C L s^2 + C R s + 1)/(C s)\, I(s)$
- $I(s)/V(s) = C s / (C L s^2 + C R s + 1)$

EXAMPLE

Ex: Consider an RL circuit with R=4, L=1/2. Find i(t) if v(t)=12u(t).



$$L\frac{di(t)}{dt}+Ri(t)=v(t)$$

$$0.5\frac{di(t)}{dt}+4i(t)=v(t)$$

$$U\sin g\_Laplace:$$

$$(0.5s+4)I(s)=V(s)$$

$$H(s)=I(s)/V(s)=\frac{1}{0.5s+4}\Rightarrow I(s)=H(s).V(s)$$

$$v(t)=12u(t)\leftrightarrow V(s)=12/s$$

$$I(s)=H(s).V(s)=\frac{24}{s(s+8)}=\frac{k_1}{s}+\frac{k_2}{s+8}$$

$$k_1=[(s-p_1)I(s)]\,\Big|_{p_1=0}=3$$

$$k_2=[(s-p_2)I(s)]\,\Big|_{p_2=-8}=-3$$

$$\Rightarrow I(s)=\frac{3}{s}+\frac{-3}{s+8}\Leftrightarrow i(t)=3u(t)-3e^{-8t};t>0$$