PROBABILITY, SIGNALS & SYSTEMS

BY: RUAA SHALLAL ANOOZ

RANDOM EXPERIMENT

- **Experiment** : any process or procedure for which more than one outcome is possible.
- **Random Experiment:** is a process by which we observe something uncertain.
- **Outcome:** is a result of a random experiment.
- **Sample Space (*S*):** is a set consisting of all of the possible experimental outcomes.

EXAMPLES

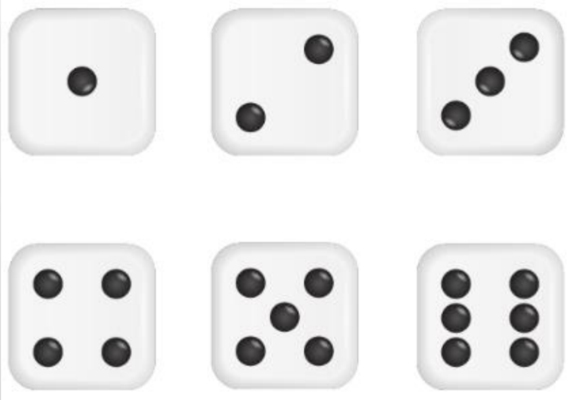
- Random Experiment toss a coin; the sample space is

S={head, tail} or {H,T}

- A die toss, what are the results of this experiment? Or what is the sample space of this experiment?

Sol: the results are or the sample space is

S={1,2,3,4,5,6}



EXAMPLES

- If two dice are rolled (or, equivalently, if one die is rolled twice), the sample space is shown in Figure 1.2.

S=$sample\ space\ outcomes^{number\ of\ iterations}$

= 6^2=36 outcomes

FIGURE 1.2 Sample space for rolling two dice

S

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

- We toss a coin three times, the sample space is :

S=2^3 = 8 outcomes

S={(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)}

EVENT

- An **event** $\boldsymbol{A}$ is a subset of the sample space $\boldsymbol{S}$. The probability of an event A, $P(A)$, is obtained by summing the probabilities of the outcomes contained within the event $\boldsymbol{A}$.

- An **event** is said to **occur** if one of the outcomes contained within the event occurs.

FOR EXAMPLE, ROLL ONE DIE.

EVEN={2, 4, 6} ODD= {1, 3, 5}

GREATER THAN 6=EMPTY SET=ϕ

- Event $\longleftrightarrow$ Set
- Sample space $\longleftrightarrow$ Universal Set
- Outcome $\longleftrightarrow$ Element

COMPLEMENTS OF EVENTS

- **Complements of Events**

The event A', the **complement** of event ***A***, is the event consisting of everything in the sample space ***S*** that is not contained within the event ***A***. In all cases

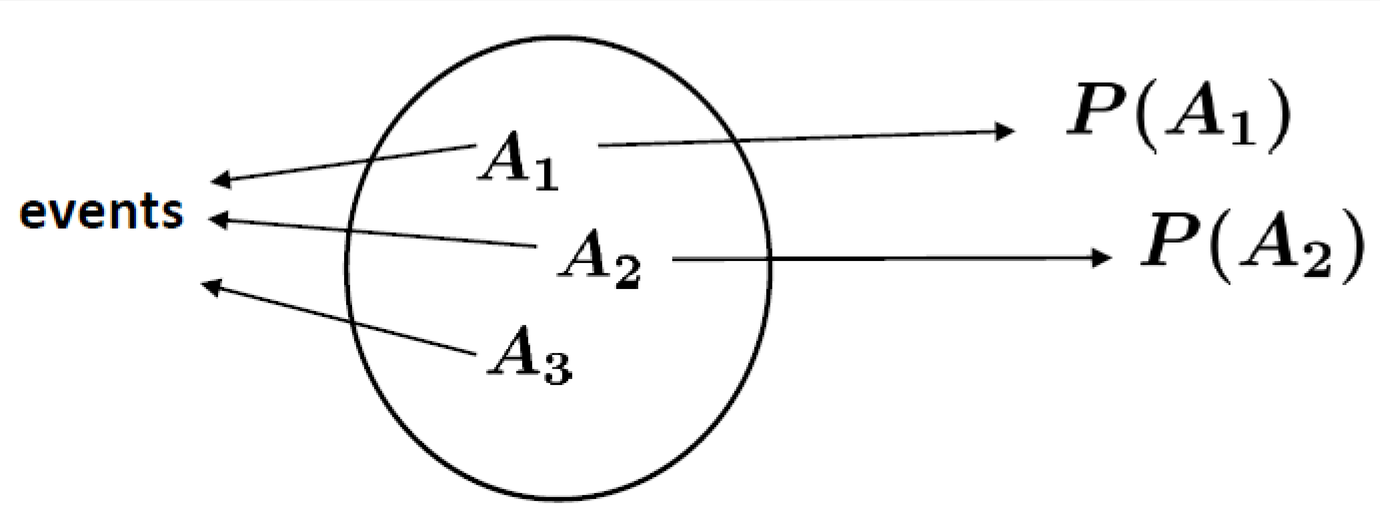
$$P(A) + P(A') = 1$$

- Events that consist of an individual outcome are sometimes referred to as **elementary events** or **simple events**

PROBABILITY

- The probability is the measure of the occurrence of an event.

$$P(\text{event}) = \frac{\textit{the number of the event occure}}{\textit{the sample space}}$$



EXAMPLE

Example: you roll a fair die. What is the probability of A={1,5}?

Sol: the die is fair, which means that all six possible outcomes are equally likely, i.e.,

$$P(\{1\})=P(\{2\})=\ldots=P(\{6\})=\frac{1}{6}$$

$$P(A)=P(\{1\})+P(\{5\})$$

$$=\frac{2}{6}=\frac{1}{3}$$

EXAMPLE

- **Example:-** two fair dice, are thrown find the following:

- A = { the sum of the scores of two dice is equal to 6 }
  = { (1,5), (2,4), (3,3), (4,2), (5,1) }

A sum of 6 will be obtained with two fair dice roughly 5 times out of 36 on average, that is, on about 14% of the throws.

$$P(A) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{5}{36}$$

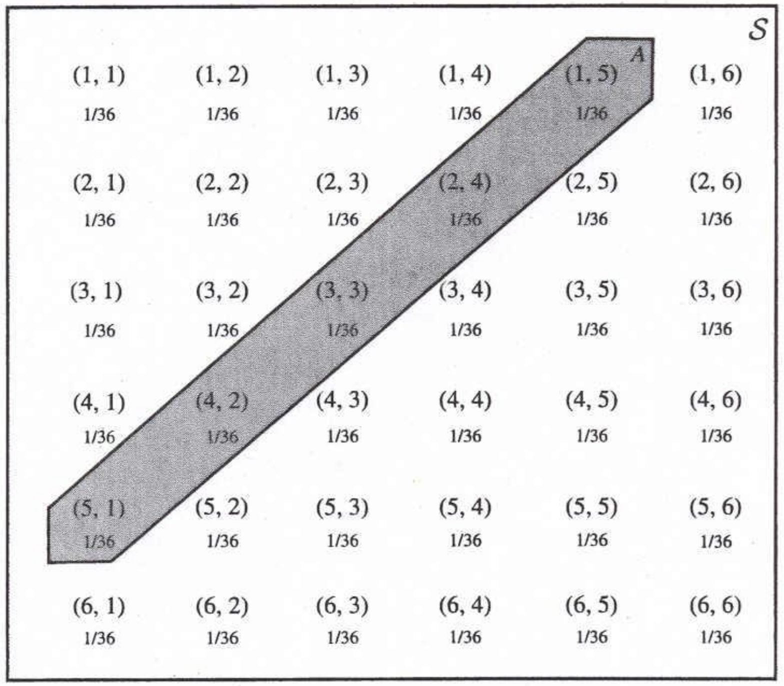
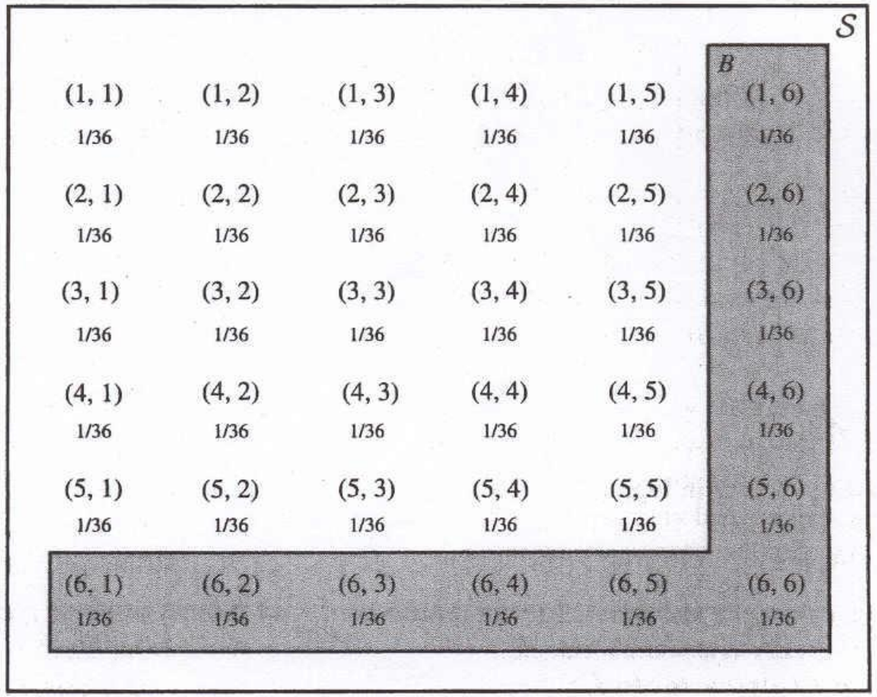


FIGURE 1.18 ●
Event A: sum equal to 6

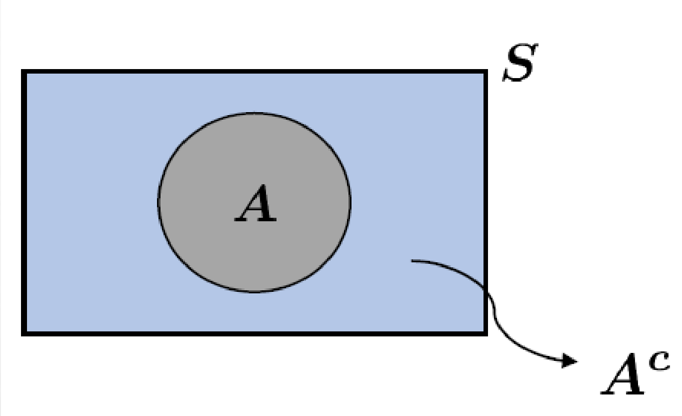
EXAMPLE

- B = { at least one of the two dice records a 6 }
  = { (1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,5), (6,4), (6,3), (6,2), (6,1) }

FIGURE 1.19 ●
Event *B*: at least one 6 recorded



AXIOMS OF PROBABILITY



- Property 1. $0 \leq Pr(A) \leq 1$
- Property 2. $P(S) = 1$
- Property 3. $P(A) + P(\bar{A}) = P(S) \Longrightarrow P(\bar{A}) = 1 - P(A)$
- Property 4. $P(A \text{ or } B) = P(A) + P(B)$ when A and B are disjoint or mutually exclusive.

AXIOMS OF PROBABILITY

$P(\emptyset) = 0,$

$\emptyset$: empty

$$P(\emptyset) = P(S^c) = 1 - P(S) = 1 - 1 = 0.$$

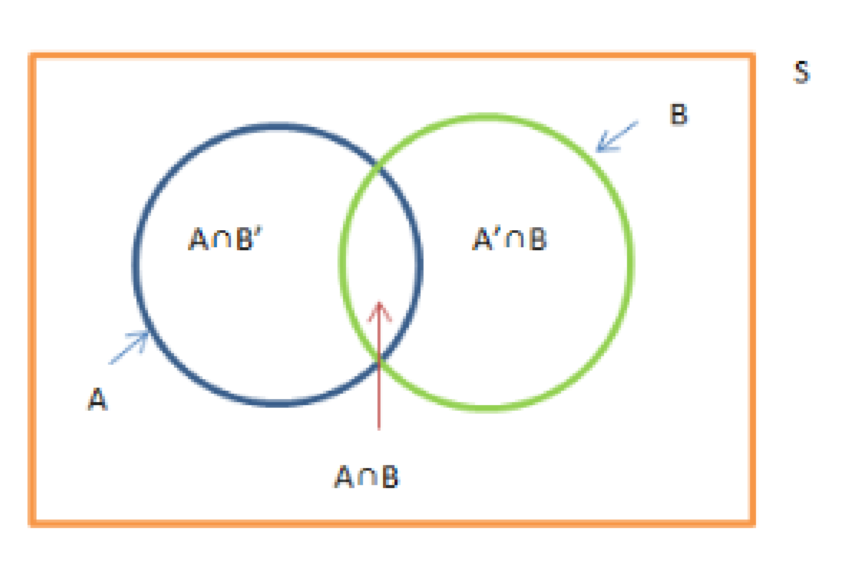
AXIOMS OF PROBABILITY

$$P(A - B) = P(A) - P(A \cap B)$$

$$A = (A \cap B) \cup (A - B)$$

$(A \cap B)$ **and** $(A - B)$ **are disjoint.**

$$P(A) = P((A \cap B) \cup (A - B))$$
$$= P(A \cap B) + P(A - B).$$



COMBINATIONS OF EVENTS

- **Intersections of Events**

The event $A \cap B$ is the **intersection** of the events ***A*** and ***B*** and consists of the outcomes that are contained within both events ***A*** and ***B***. The probability of this event, $P(A \cap B)$, is the probability that both events ***A*** and ***B*** occur **simultaneously**.

- A sample space S consists of 9 outcomes

$$P(A) = 0.01 + 0.07 + 0.19 = 0.27$$

$$P(B) = 0.07 + 0.19 + 0.04 + 0.14 + 0.12 = 0.56$$

$$P(A \cap B) = 0.07 + 0.19 = 0.26$$

FIGURE 1.26 • Events A and B

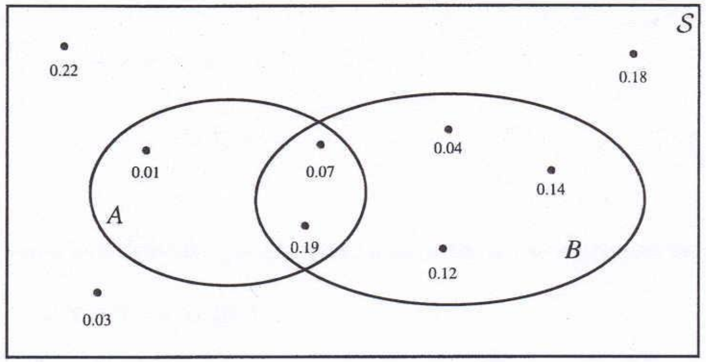
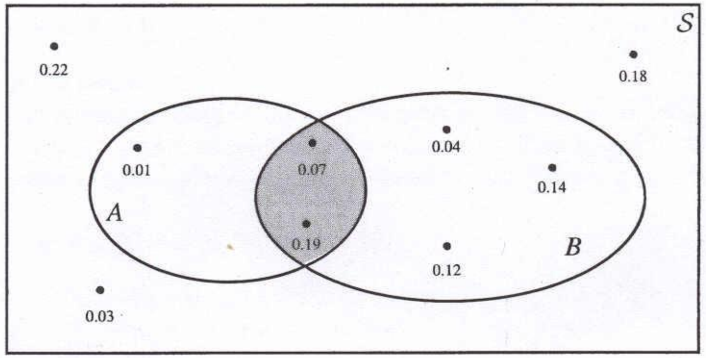


FIGURE 1.27 • The event $A \cap B$



INTERSECTIONS OF EVENTS

FIGURE 1.28 ● The event $A' \cap B$

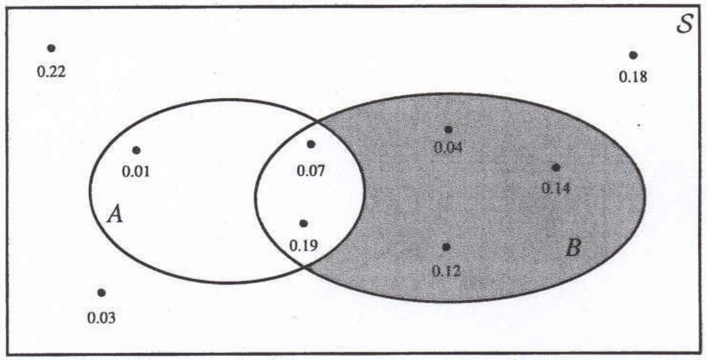
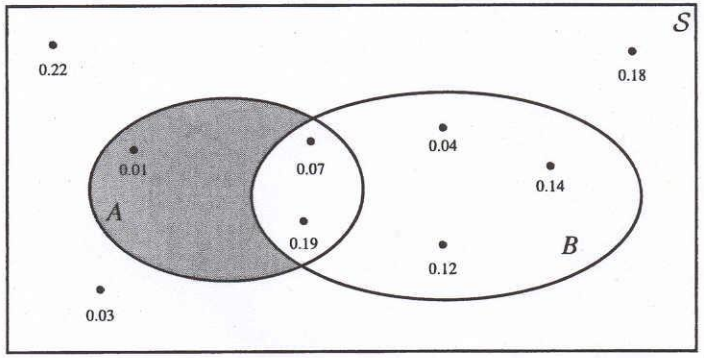


FIGURE 1.29 ● The event $A \cap B'$



$$P(A' \cap B) = 0.04 + 0.14 + 0.12 = 0.30$$

$$P(A \cap B') = 0.01$$

$$P(A \cap B) + P(A \cap B') = 0.26 + 0.01 = 0.27 = P(A)$$

$$P(A \cap B) + P(A' \cap B) = 0.26 + 0.30 = 0.56 = P(B)$$

INTERSECTIONS OF EVENTS

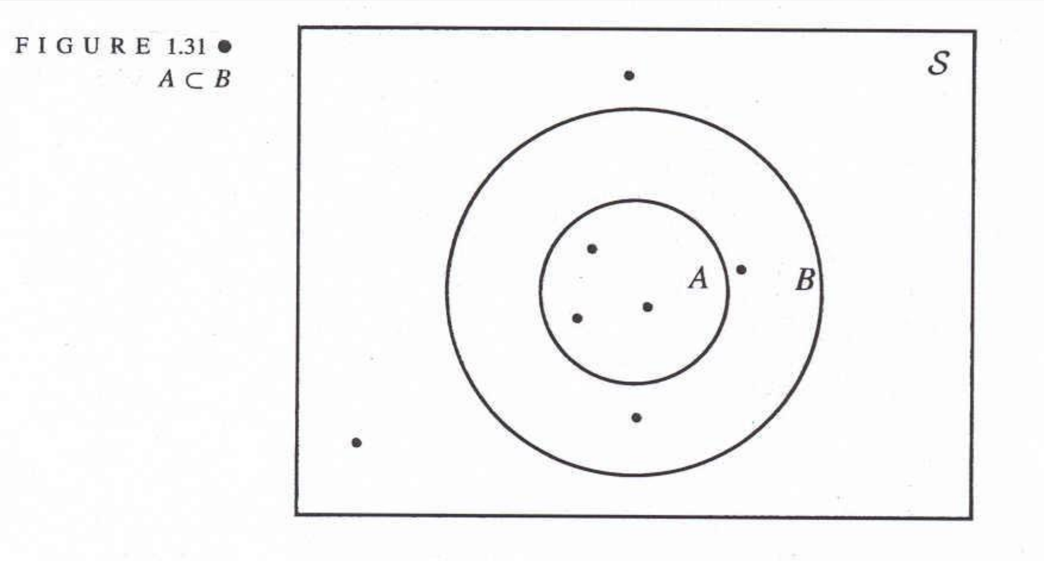
□ $P(A \cap B) + P(A \cap B') = P(A)$

$P(A \cap B) + P(A' \cap B) = P(B)$

- **Mutually Exclusive Events**

Two events ***A*** and ***B*** are said to be **mutually exclusive** if $A \cap B = \varnothing$ so that they have **no outcomes in common**.

INTERSECTIONS OF EVENTS



$$A \cap B = A$$

- $A \cap B = B \cap A$

$$A \cap A = A$$

$$A \cap S = A$$

$$A \cap \varnothing = \varnothing$$

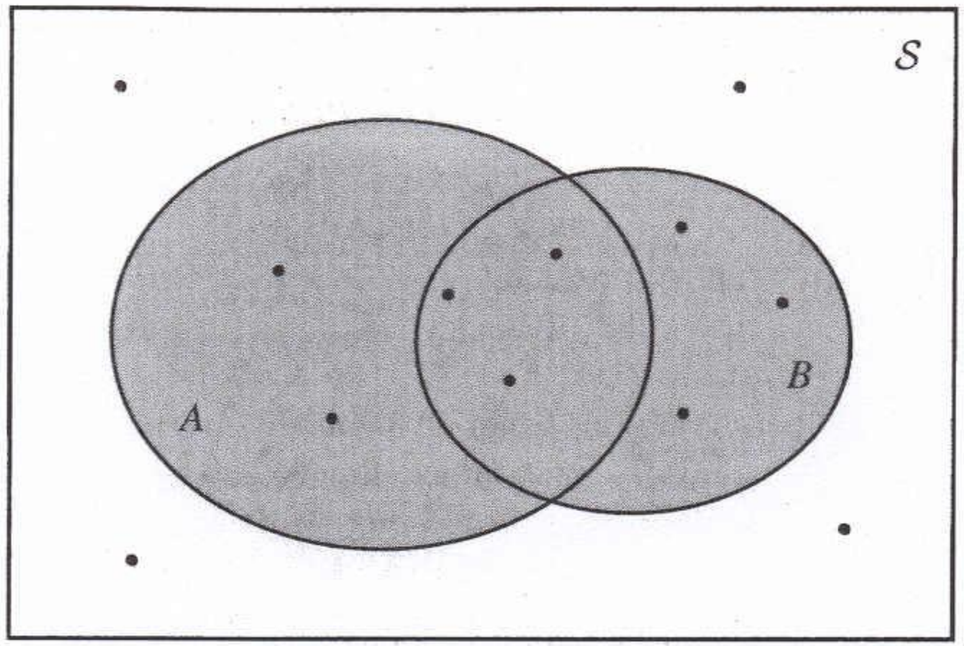
$$A \cap A' = \varnothing$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

UNIONS OF EVENTS

- The event $A \cup B$ is the **union** of events ***A*** and ***B*** and consists of the outcomes that are contained within at least one of the events ***A*** and ***B***. The probability of this event, $P(A \cup B)$, is the probability that at least one of the events ***A*** and ***B*** occurs.

FIGURE 1.32 • The event $A \cup B$

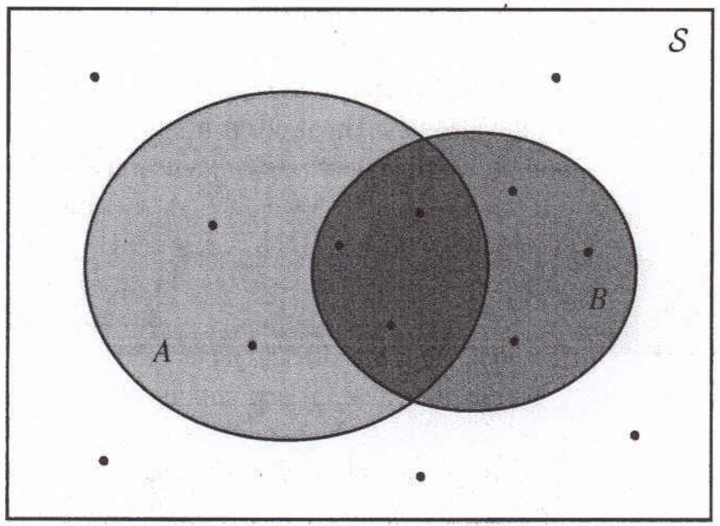


UNIONS OF EVENTS

- Notice that the outcomes in the event $A \cup B$ can be classified into three kinds:

1. in event A but not in event B
2. in event B but not in event A
3. in both events A and B

FIGURE 1.33
Decomposition of the event $A \cup B$



$$P(A\cup B) = P(A\cap B') + P(A'\cap B) + P(A\cap B)$$

UNIONS OF EVENTS

- $P(A \cap B') = P(A) - P(A \cap B)$

  $P(A' \cap B) = P(B) - P(A \cap B)$

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- If the events A and B are mutually exclusive so that

  $P(A \cap B) = 0$, then $P(A \cup B) = P(A) + P(B)$

UNIONS OF EVENTS

- Simple results concerning the unions of events

$$\left.\begin{aligned}(A\cup B)' &= A'\cap B' \\ (A\cap B)' &= A'\cup B'\end{aligned}\right\}$$ **(Demorgan's law)**

$$A\cup B = B\cup A$$

$$A\cup A = A$$

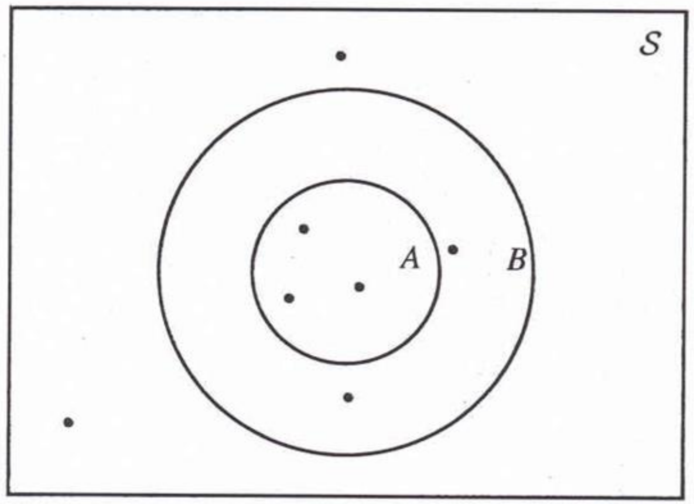
$$A\cup S = S$$

$$A\cup\varnothing = A$$

$$A\cup A' = S$$

$$A\cup(B\cup C) = (A\cup B)\cup C$$

FIGURE 1.31 ● $A\subset B$



EXAMPLE

- **Television Set Quality**

A company that manufactures television sets performs a final quality check on each appliance before packing and shipping it.

The quality check has an evaluation of the quality of the **picture** and the **appearance**. Each of the two evaluations is graded as *Perfect (P)*, *Good (G)*, *Satisfactory (S)*, or *Fail (F)*. Find the probability of the following:

S

(P, P) 0.140	(P, G) 0.102	(P, S) 0.157	(P, F) 0.007
(G, P) 0.124	(G, G) 0.141	(G, S) 0.139	(G, F) 0.012
(S, P) 0.067	(S, G) 0.056	(S, S) 0.013	(S, F) 0.010
(F, P) 0.004	(F, G) 0.011	(F, S) 0.009	(F, F) 0.008

FIGURE 1.38 ●
Probability values for television set example

EXAMPLE

A= An appliance that fails on either of the two evaluations and that score an evaluation of *Satisfactory* on **both** accounts will **not be shipped**.

A = { an appliance cannot be shipped }

= { *(F,P), (F,G), (F,S), (F,F), (P,F), (G,F), (S,F), (S,S)* }

P(A) = 0.074

About 7.4% of the television sets will fail the quality check.

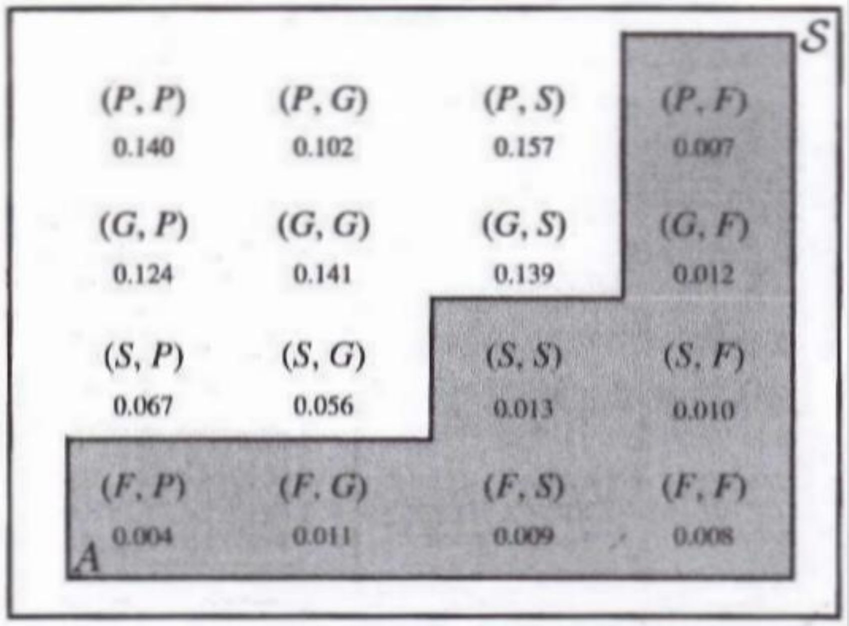


FIGURE 1.39 • Event *A*: appliance not shipped

EXAMPLE

B = { picture satisfactory or fail }

P(B) = 0.178

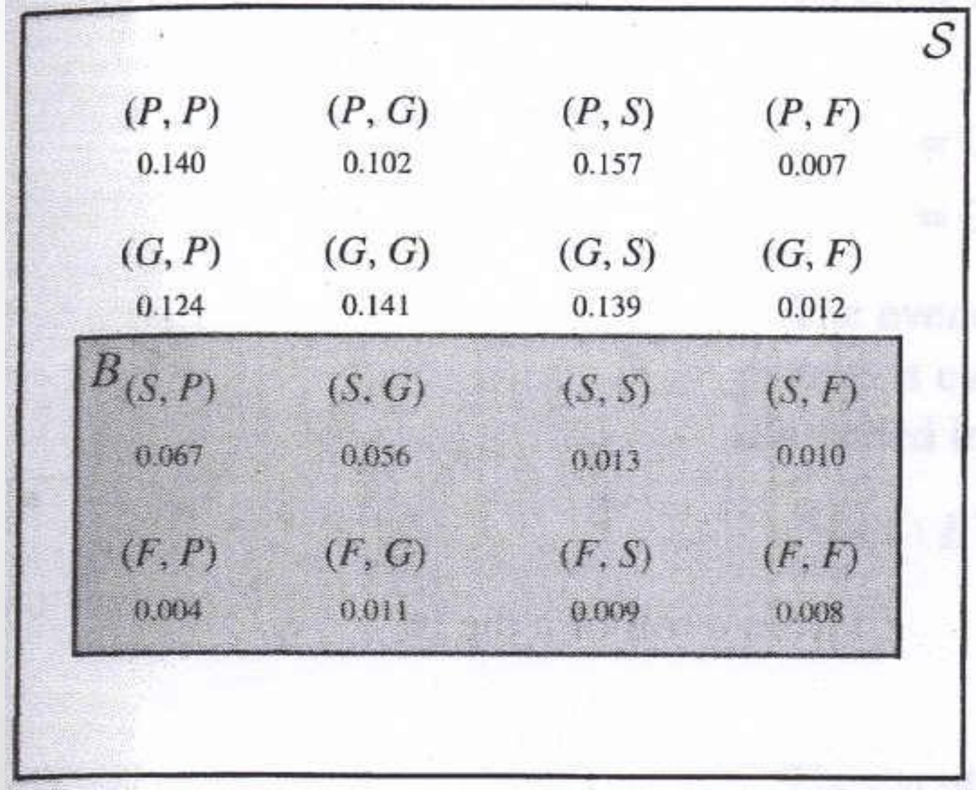


FIGURE 1.40 ●
Event B: picture satisfactory or fail

EXAMPLE

$A \cap B$ = { Not shipped **and** the picture satisfactory or fail }

$$P(A \cap B) = 0.055$$

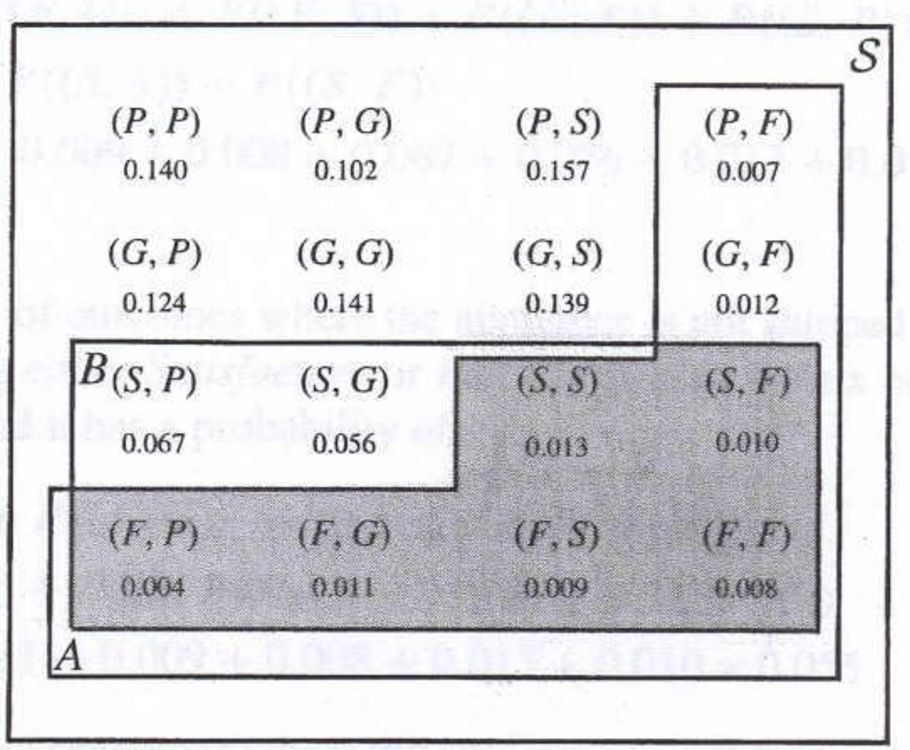


FIGURE 1.41 •
Event $A \cap B$

EXAMPLE

$A \cup B$ = { the appliance was **either** not shipped **or** the picture was evaluated as being either *Satisfactory* or *Fail* }

$$P(A \cup B) = 0.197$$

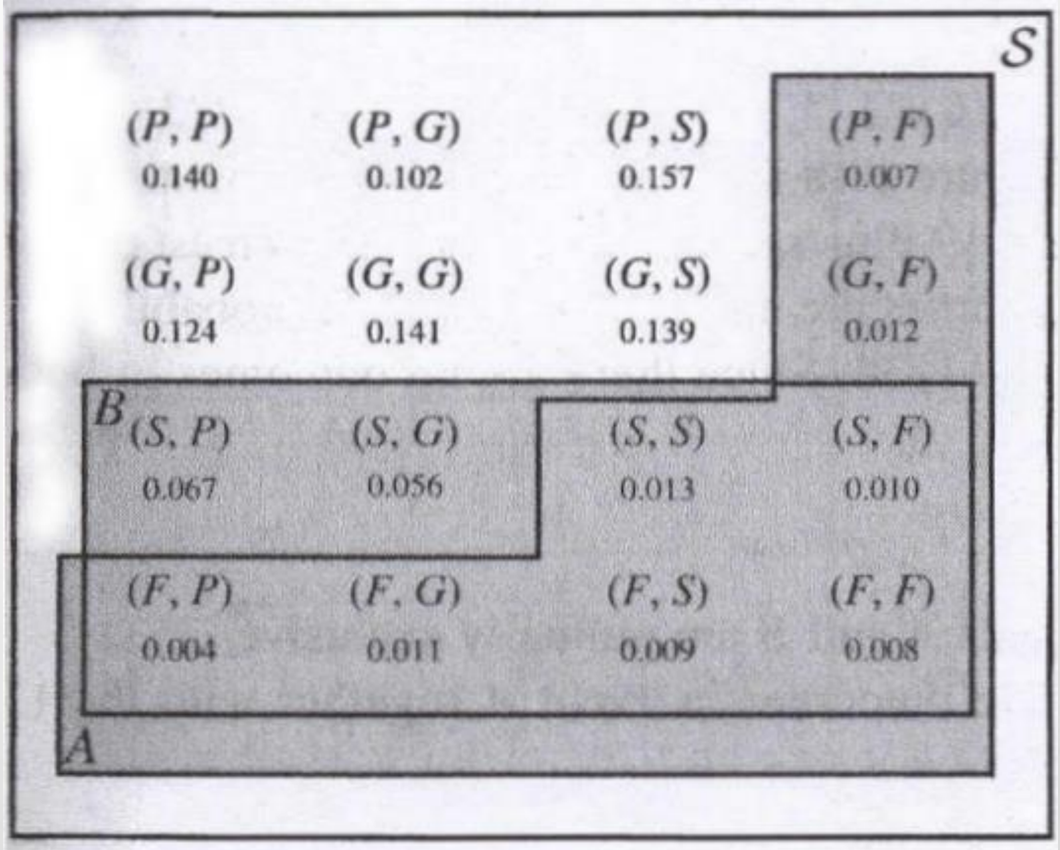


FIGURE 1.42 •
Event $A \cup B$

EXAMPLE

$P(A \cap B')$ = { Television sets that have a picture evaluation of either *Perfect* or *Good* but that cannot be shipped }

$$P(A \cap B') = 0.019$$

Notice

$$P(A \cap B) + P(A \cap B') = 0.055 + 0.019$$
$$= 0.074$$
$$= P(A)$$

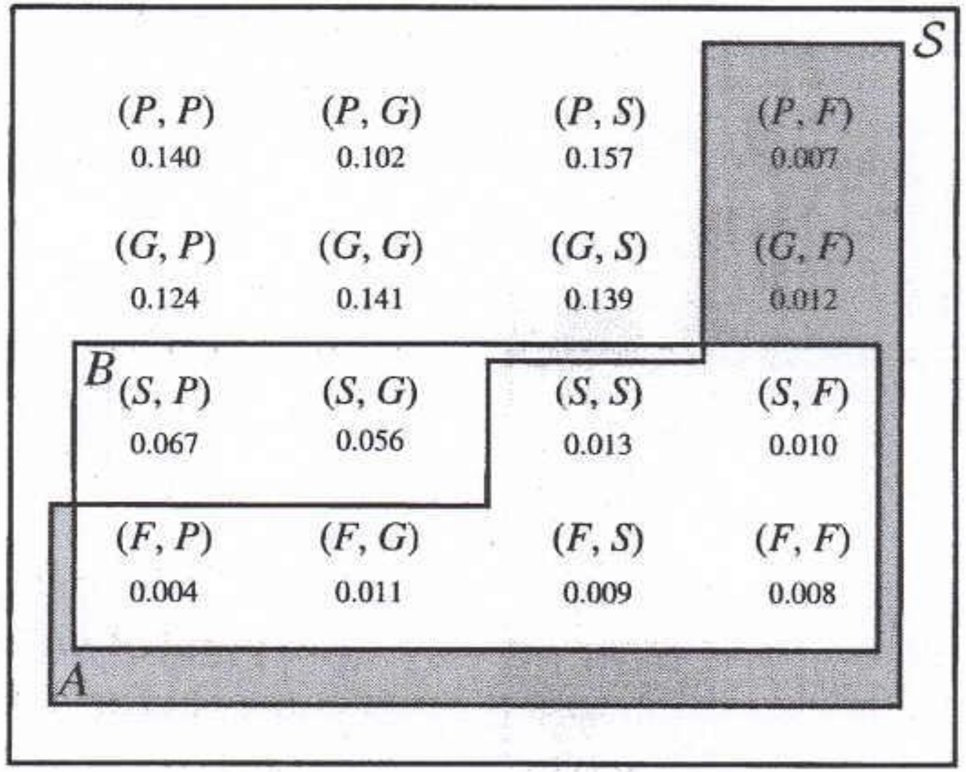


FIGURE 1.43 ●
Event $A \cap B'$

EXAMPLE

Example:- One die rolling, Find the following:

A = { an even score is obtained from a roll of a die }

B = {the numbers that are greater than 3}

A = { 2, 4, 6 } ***B*** = { 4, 5, 6 }

then

$$A \cap B = \{4,6\} \text{ and } A \cup B = \{2,4,5,6\}$$

$$P(A \cap B) = \frac{2}{6} = \frac{1}{3} \text{ and } P(A \cup B) = \frac{4}{6} = \frac{2}{3}$$

EXAMPLE

Example:- For two dice rolling, find the following

A = { the sum of the scores is equal to 6 } ,

A= { (1,5), (2,4), (3,3), (4,2), (5,1) }

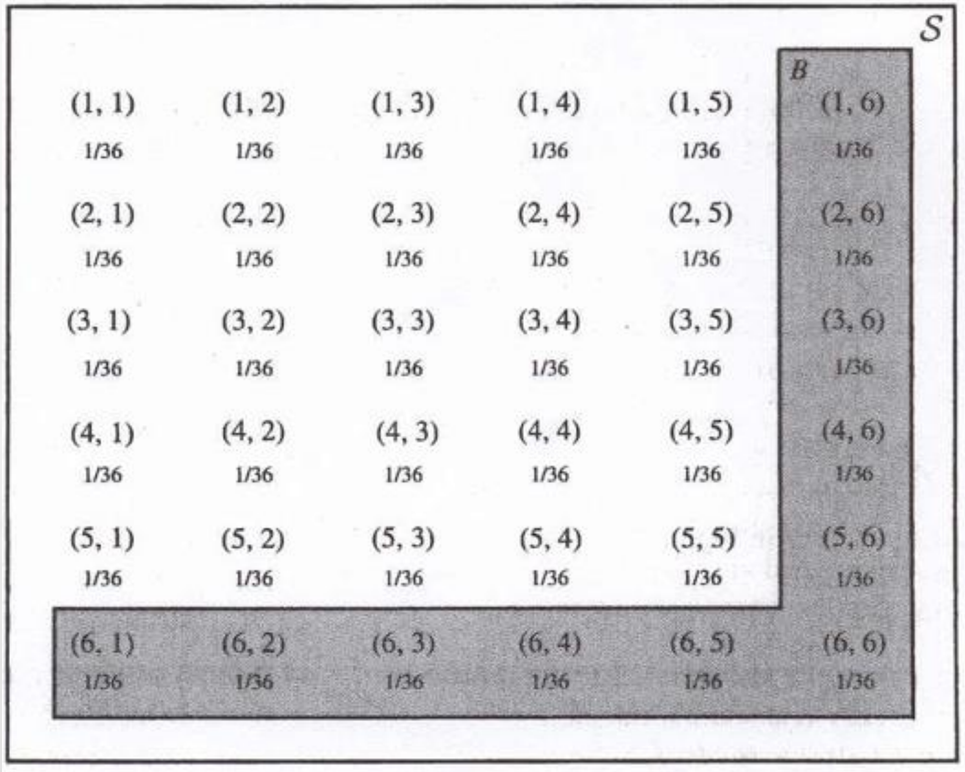
B = { at least one of the two dice records a 6 }

P(A) = 5/36 and ***P(B) = 11/36***

A and ***B*** are mutually exclusive=>

$A \cap B = \varnothing$ and $P(A \cap B) = 0$

$$P(A \cup B) = \frac{16}{36} = \frac{4}{9} = P(A) + P(B)$$

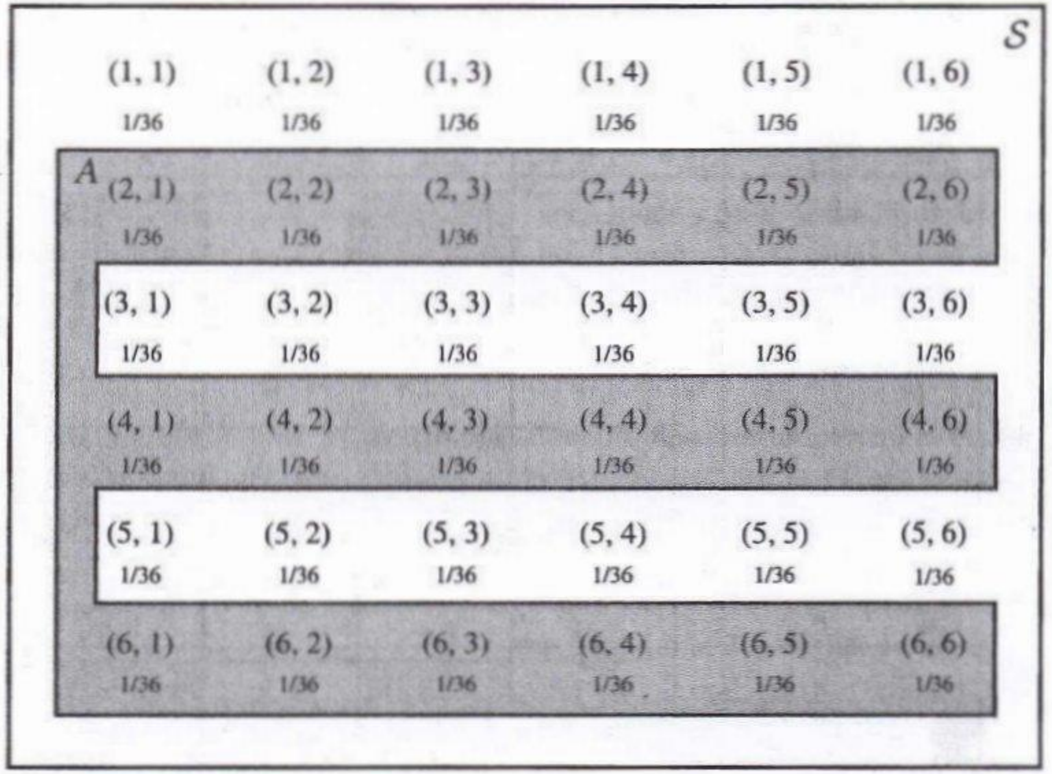


EXAMPLE

- One die is red and the other is blue ⟶ (red, blue).

A = { an even score is obtained on the red die }

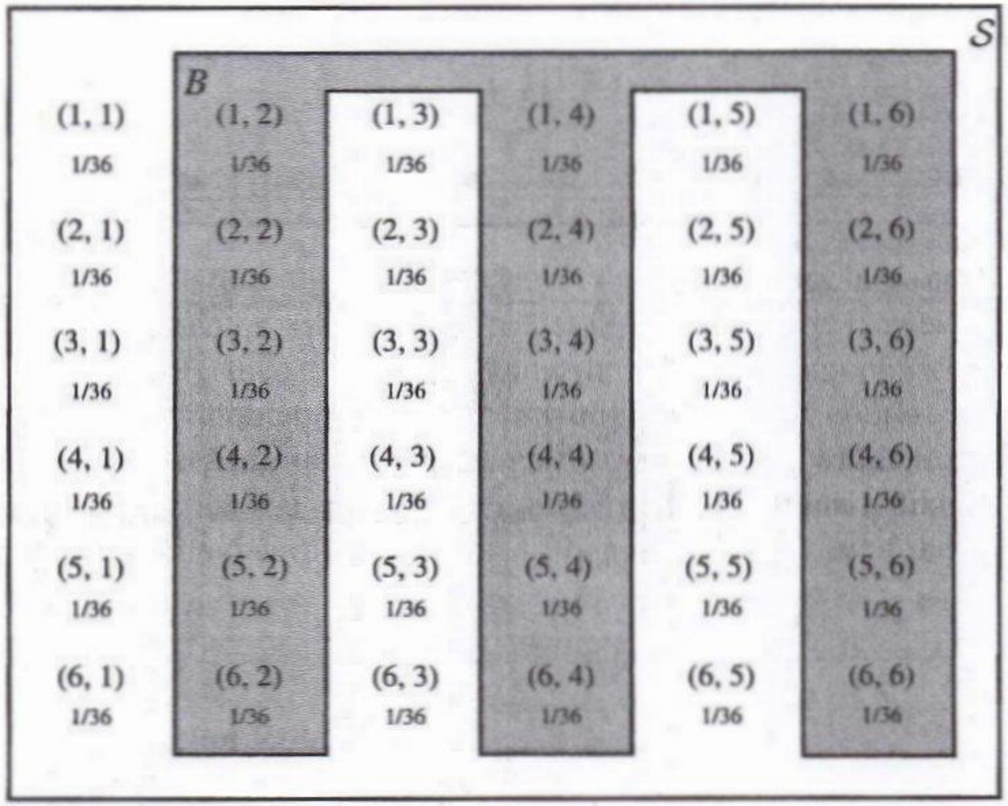
FIGURE 1.44
Event A: even score on red die



EXAMPLE

B = { an even score is obtained on the blue die }

FIGURE 1.45 •
Event *B*: even score on blue die

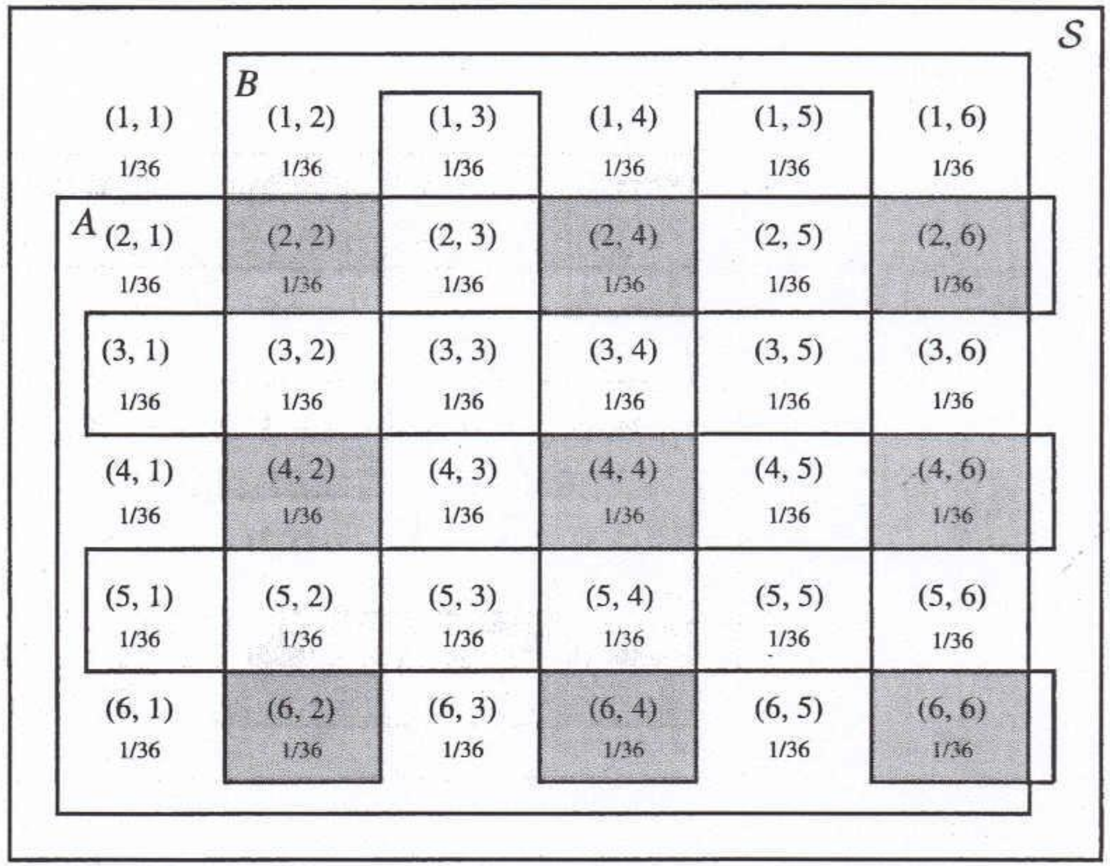


EXAMPLES

$A \cap B$ = { both dice have even scores }

$$P(A \cap B) = \frac{9}{36} = \frac{1}{4}$$

FIGURE 1.46 • Event $A \cap B$



EXAMPLES

$A \cup B$ = { at least one die has an even score }

$$P(A \cup B) = \frac{27}{36} = \frac{3}{4}$$

FIGURE 1.47 ● Event $A \cup B$