PROBABILITY, SIGNALS & SYSTEMS

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COMBINATIONS OF THREE OR MORE EVENTS

- Union of Three Events

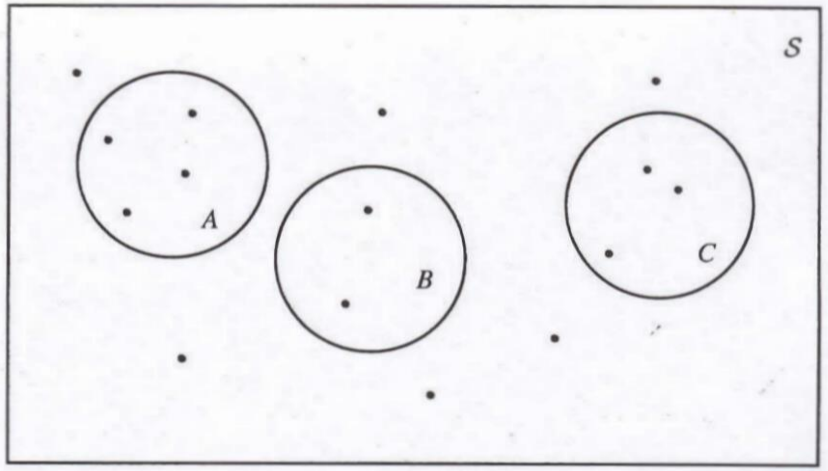
$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

If $A \cap B = \varphi$

$A \cap C = \varphi$

$C \cap B = \varphi$

FIGURE 1.52 ●
Three mutually exclusive events

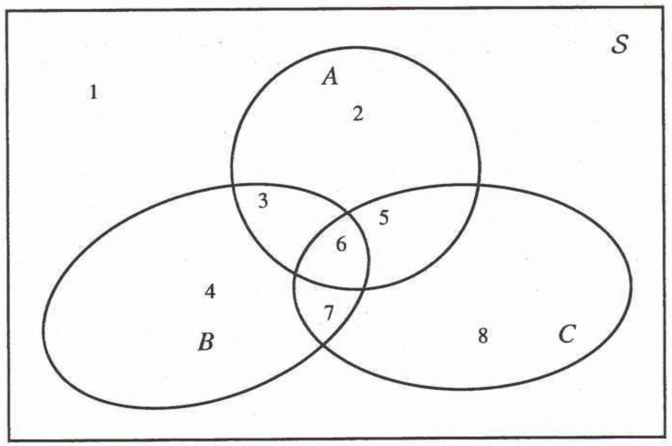


COMBINATIONS OF THREE OR MORE EVENTS

- Union of Three Events

$$
\begin{aligned}
P(A\cup B\cup C) = {} & [P(A)+P(B)+P(C)] \\
& -[P(A\cap B)+P(A\cap C)+P(B\cap C)] \\
& +P(A\cap B\cap C)
\end{aligned}
$$

FIGURE 1.51 ●
Three events decompose sample space into eight regions



EXAMPLE

- Assume the following information is correct:
- The chance of it raining today is 50%.
- The chance it will rain tomorrow is 40%.
- The probability that it will not rain today and not tomorrow is 30%.

Find the probabilities for the following events:

1. It will rain today or tomorrow?
2. It will rain today and tomorrow?
3. It will rain today, but not tomorrow?
4. It will rain today or tomorrow, but not on both days?

EXAMPLE

Sol:

P(A) = 50% = 0.5, P(B) = 40% = 0.4, $P(A^c \cap B^c)$ = 30% = 0.3

1. $A^c \cap B^c = (A \cup B)^c = 0.3$

And from the definition of complement we see that: $A + A^c = S$ ⟵ **(Sample space)**

When take the probability for both side we get: $p(A) + p(A^c) = p(S)$ **= 1**

$\therefore P(A) + p(A^c) = 1$

$P(A \cup B) = 1 - (A \cup B)^c = 1 - 0.3 = 0.7$

2. We will use of inclusion-exclusion principle

$P(A \cup B) = p(A) + p(B) - P(A \cap B) \Rightarrow P(A \cap B) = p(A) + p(B) - P(A \cup B)$

$\therefore P(A \cap B) = 0.5 + 0.4 - 0.7 = 0.2$

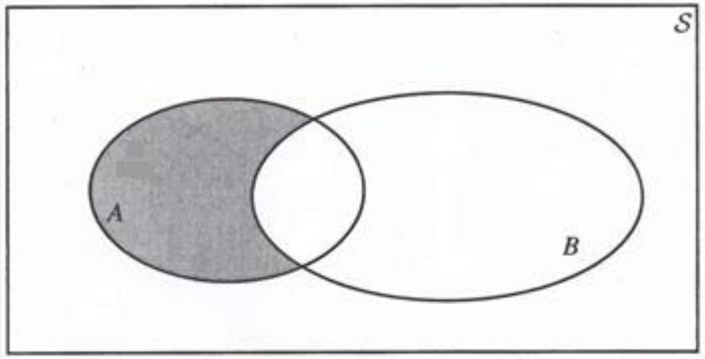
EXAMPLE

Sol:

3. $A \cap B^c = p(A) - P(A \cap B)$

$= 0.5 - 0.2 = 0.3$

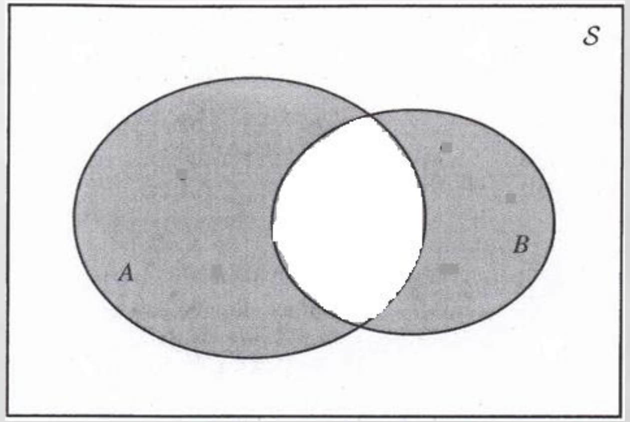
This match that **P(A-B)**



4. $P(B-A) = p(B) - P(A \cap B)$

$= 0.4 - 0.2 = 0.2$

$P((A-B) \cup (B-A)) = 0.3+0.2 = 0.5$



EXAMPLE

Roll a die twice and observed x_1 and x_2. Find:

1. S.
2. A: $x_1 + x_2 = 4$, Find the elements in A, and P(A).
3. B: $x_1 + x_2 = 6$ or 7, Find P(B).

FIGURE 1.2 ●
Sample space for rolling two dice

S

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Sol:

1. S = 6^2= 36.
2. A={(1,3), (2,2), (3,1)}, P(A) = $\frac{3}{36} = \frac{1}{12}$
3. B={(1,5), (1,6), (2,4), (2,5), (3,3), (3,4), (4,2), (4,3), (5,1), (5,2), (6,1)}

   P(B) = $\frac{11}{36}$

CONTINUOUS PROBABILITY SPACE

Example: I choose a point completely at random in [0,1].

a) $P([0, 0.5]) = 0.5$

b) $P([0, 0.25]) = 0.25$

c) $P([a, b]) = b - a, \quad 0 \leq a \leq b \leq 1$

d) $P(\{0.5\}) = 0 = P([0.5, 0.5]) = 0.5 - 0.5 = 0$

CONTINUOUS PROBABILITY SPACE

Key point: Axioms of Probability applies to continuous probability spaces.

Example: Suppose we know that the probability that a certain machine lasts more than or equal to x years is :

$$P(T \geq x) = \frac{1}{2^x}$$

where

$$T: \text{ Lifetime}$$

Find the following sets:

a) $P(T \geq 1)$

b) $P(T \geq 2)$

c) $P(1 \leq T \leq 2)$

sol:

a) $P(T \geq 1) = \frac{1}{2^1} = \frac{1}{2}$

b) $P(T \geq 2) = \frac{1}{2^2} = \frac{1}{4}$

c) $P(1 \leq T \leq 2) = P(T \geq 1) - P(T \geq 2) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

CONDITIONAL PROBABILITY

- A conditional probability contains a condition that may limit the sample space for an event. The probability that an event B will occur given that another event, A, has already occurred the two events must be dependent

$$P\left(B \middle| A\right) = \frac{P\left(A \text{ and } B\right)}{P\left(A\right)}$$

Read | as "given that" or "under the condition that"

$$\frac{|A \cap B|}{|B|} = \frac{\frac{|A \cap B|}{|S|}}{\frac{|B|}{|S|}} = \frac{P(A \cap B)}{P(B)}$$

CONDITIONAL PROBABILITY

- To find the conditional probability P(A | B), use the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- The conditional probability P(B | A) is given by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

EXAMPLE

- Roll a fair die, what is the probability that the outcome is an even number given it was less than or equal to 3, i.e.,

Sol: S={1,2,3,4,5,6}

Let A={2,4,6} and B={1,2,3}

$P(A \cap B)$={ 2}

$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1}{3}$

CONDITIONAL PROBABILITY

- Conditional probability satisfies the probability axioms :

a) For any event $A, \; P(A|B) \geq 0$

b) Conditional probability of B given B is $P(B|B) = 1$

c) If $A_1, A_2, A_3, \cdots$ are disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cdots |B) = P(A_1|B) + P(A_2|B) + P(A_3|B) + \cdots .$$

EXAMPLE

- Roll two dice x_1, x_2,

A: 3 dots are shown at least on one die,

B: $x_1 + x_2 = 6$ Find $P(A|B)$

SOL: A be either x_1=3 or x_2=3

A={(1,3),(2,3),(3,3),(3,1),(3,2), (3,4),(3,5),(3,6),(4,3),(5,3),(6,3)}

B={(1,5),(2,4),(3,3),(4,2),(5,1)}

$P(A \cap B) = \{\ (3,3)\}$ $\qquad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{5}$

FIGURE 1.2
Sample space for rolling two dice

$\mathcal{S}$

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

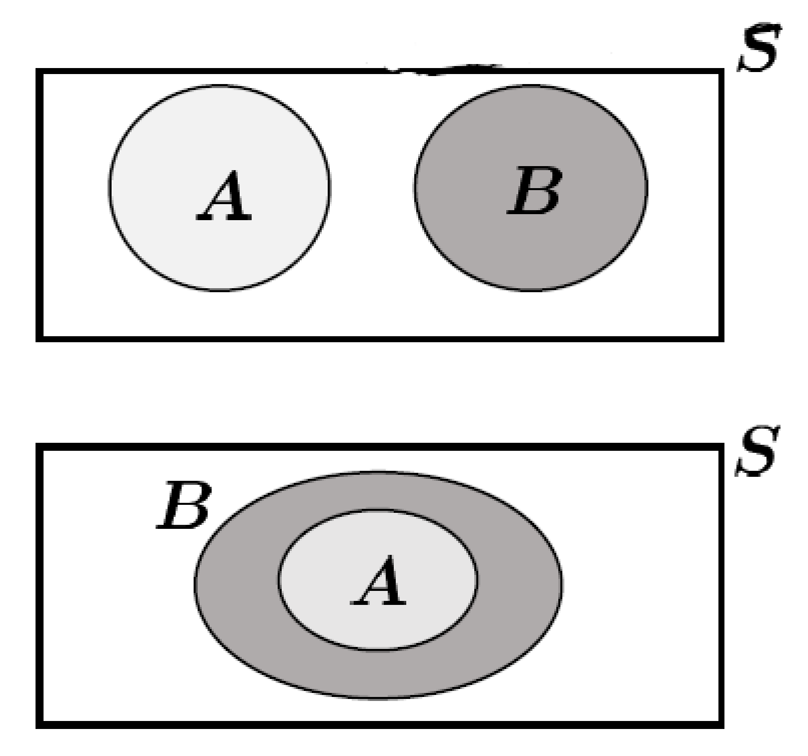
CONDITIONAL PROBABILITY SPECIAL CASES

1) A and B are disjoint:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0$$

2) A ⊂ B

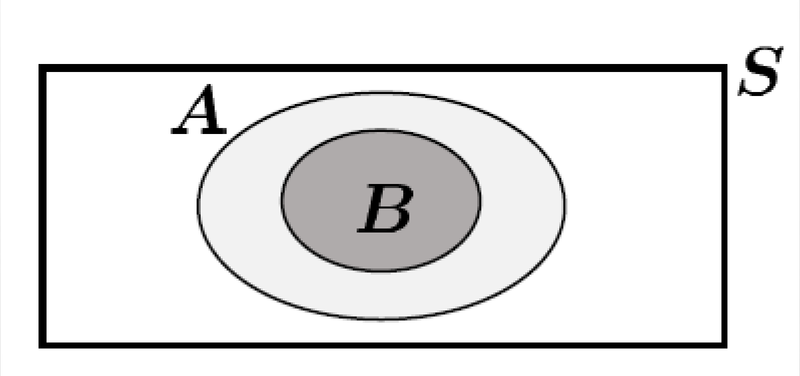
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$



CONDITIONAL PROBABILITY SPECIAL CASES

3) B ⊂ A

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$



EXAMPLE

- Roll a die, what is the probability that it is larger than or equal to 5, given that it is an even number ?

SOL: S={1,2,3,4,5,6}   A={5,6}   B={2,4,6} $P(A \cap B)$ ={6}

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{3}$$

INDEPENDENT EVENTS

Events are **independent events** if the occurrence of one event does not affect the probability of the other. In another word, Two events A and B are independent if and only if:

$$P(A|B) = P(A)$$

Probability of Independent Events

If A and B are independent events, then $P(A \text{ and } B) = P(A) \cdot P(B)$.

$$P(A|B)\,P(B) = P(A)\,P(B) \longrightarrow P(A \cap B) = P(A)\,P(B)$$

INDEPENDENT EVENTS

Notes:

Disjoint (mutually exclusive) ≠ Independent

Disjoint: $A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$

Independent: $P(A \cap B) = P(A).P(B)$, $P(A|B) = P(A)$, $P(B|A) = P(B)$

INDEPENDENT EVENTS

Three events A, B, and C are independent if all of the following conditions hold

$$P(A \cap B) = P(A)P(B),$$

$$P(A \cap C) = P(A)P(C),$$

$$P(B \cap C) = P(B)P(C),$$

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

EXAMPLE

A six-sided cube (biased die) is labeled with the numbers 1, 2, 2, 3, 3, and 3. Four sides are colored red, one side is white, and one side is yellow. Find the probability

1. tossing 2, then 2.
2. tossing red, then white, then yellow
3. rolling a 6 on one number cube and a 6 on another number cube
4. tossing heads, then heads, and then tails when tossing a coin 3 times

EXAMPLE

Sol:

1. Tossing a 2 once does not affect the probability of tossing a 2 again, so the events are independent.

$$P(2 \text{ and then } 2) = P(2) \bullet P(2)$$

$$= 2/6 * 2/6$$

$$\frac{1}{3} \bullet \frac{1}{3} = \frac{1}{9}$$

2 of the 6 sides are labeled 2.

EXAMPLE

2. The result of any toss does not affect the probability of any other outcome.

P(red, then white, and then yellow) = P(red) • P(white) • P(yellow)

$$= \frac{4}{6} \bullet \frac{1}{6} \bullet \frac{1}{6} = \frac{4}{216} = \frac{1}{54}$$

4 of the 6 sides are red; 1 is white; 1 is yellow.

EXAMPLE

3.

$P(6 \text{ and then } 6) = P(6) \bullet P(6)$

$\frac{1}{6} \bullet \frac{1}{6} = \frac{1}{36}$ *1 of the 6 sides is labeled 6.*

4.

$P(\text{heads, then heads, and then tails}) = P(\text{heads}) \bullet P(\text{heads}) \bullet P(\text{tails})$

$\frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} = \frac{1}{8}$ *1 of the 2 sides is heads.*

DEPENDENT EVENTS

Events are **<u>dependent events</u>** if the occurrence of one event affects the probability of the other. For example, suppose that there are 2 lemons and 1 lime in a bag. If you pull out two pieces of fruit, the probabilities change depending on the outcome of the first.

EXAMPLE

- Education The table shows students by gender and by type of school in 2005. You pick a student at random. What is P(female | graduate school)?

Sol:

The condition that the person selected is at graduate school

limits the sample space to the 3,303,000 graduate students.

Of those, 1,954,000 are female.

P(female | graduate school) = 1954/3303≈ 0.59

What's the condition? The student is at a graduate school.

Student Genders

	Males (in thousands)	Females (in thousands)
Two-year colleges	1866	2462
Four-year colleges	4324	5517
Graduate schools	1349	1954

EXAMPLE

- Your neighbor has 2 children. You learn that he has a son, Joe. What is the probability that Joe's sibling is a brother?

Sol:

Consider the experiment of selecting a random family having two children and recording whether they are boys or girls. Then, the sample space is S = {BB,BG,GB,GG}, then S = {BB,BG,GB}.

We want to compute $P(BB) = \frac{1}{3}$