PROBABILITY, SIGNALS & SYSTEMS

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RANDOM VARIABLES

- In an experiment, a measurement is usually denoted by a variable such as X.

- In a random experiment, a variable whose measured value can change (from one replicate of the experiment to another) is referred to as a **random variable**.

RANDOM VARIABLES

- Random variable ≡ a numerical quantity that takes on different values depending on chance

- Two types of random variables

- Discrete random variables (countable set of possible outcomes)
- Continuous random variable (unbroken chain of possible outcomes)

TWO TYPES OF RANDOM VARIABLES

- **Discrete random variables**
  - Number of sales
  - Number of calls
  - Shares of stock
  - People in line
  - Mistakes per page
  - Number of scratches on a surface

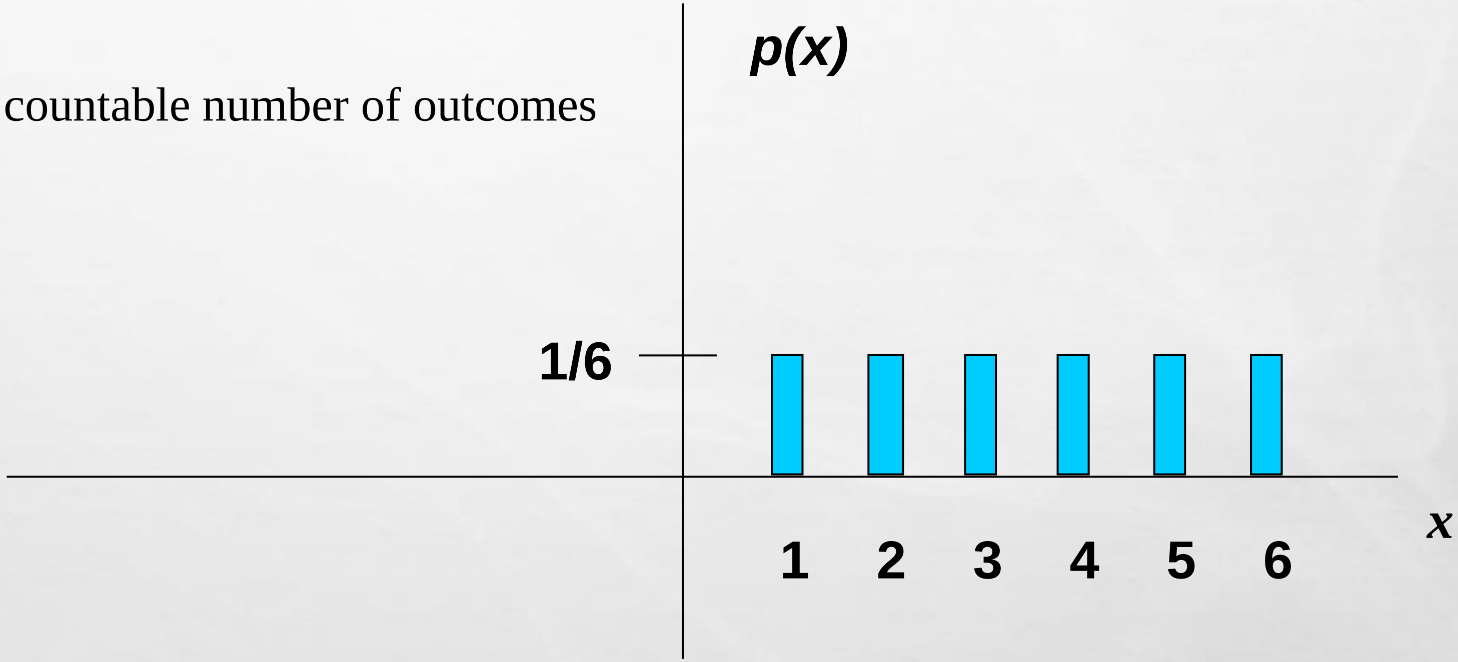
- **Continuous random variables**
  - Length
  - Depth
  - Volume
  - Time
  - Weight
  - Pressure
  - Temperature
  - Voltage
  - Electrical current

DISCRETE RANDOM VARIABLES

Discrete random variables have a countable number of outcomes

Examples: dice, counts, coins, etc.

Discrete example: roll of a die



$$\sum_{\text{all } x} P(x) = 1$$

PROBABILITY MASS FUNCTION

- **Probability Mass Function (pmf)**
- A set of probability value p_i assigned to each of the values taken by the discrete random variable x_i
  - $0 \le p_i \le 1$ and $\sum_i p_i = 1$
  - Probability : $P(X = x_i) = p_i$ for i=1,2,3,⋯

For a discrete random variable X with possible values $x_1, x_2, \ldots, x_n$, the **probability mass function** (or pmf) is

$$f(x_i) = P(X = x_i) \quad (3\text{-}13)$$

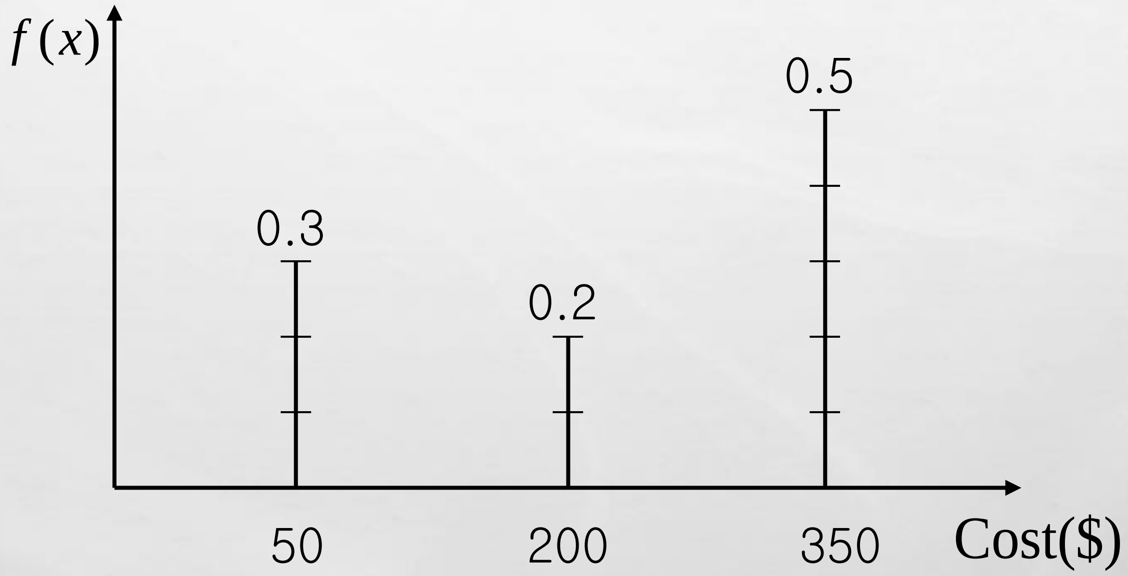
EXAMPLE

- Machine Breakdowns
  - Sample space : $S = \{electrical, mechanical, misuse\}$
  - Each of these failures may be associated with a repair cost
  - State space : $\{50, 200, 350\}$
  - Cost is a random variable : 50, 200, and 350

EXAMPLE

x_i	50	200	350
p_i	0.3	0.2	0.5

- Machine Breakdowns
  - P(cost=50)=0.3, P(cost=200)=0.2, P(cost=350)=0.5
  - 0.3 + 0.2 + 0.5 =1



EXAMPLE

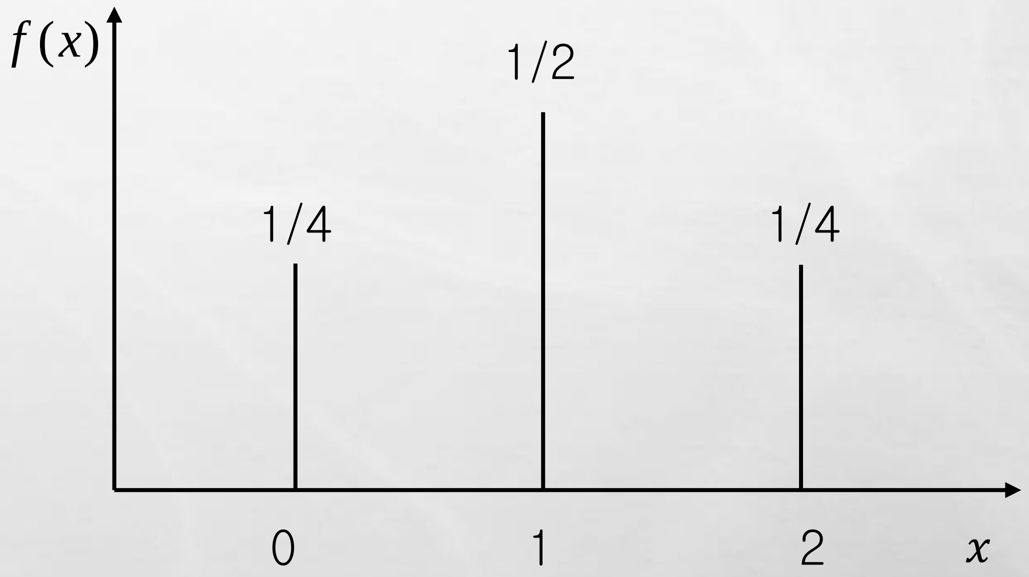
Example 1. Toss a fair coin twice, # of heads. Find the probability mass function

Solution:

S={HH, HT, TH, TT}

P{HH}=P{HT}=P{TH}=P{TT}=1/4

$$P_x(x) = \begin{cases} \frac{1}{4} & for\ x = 0 \\ \frac{2}{4} & for\ x = 1 \\ \frac{1}{4} & for\ x = 2 \end{cases}$$



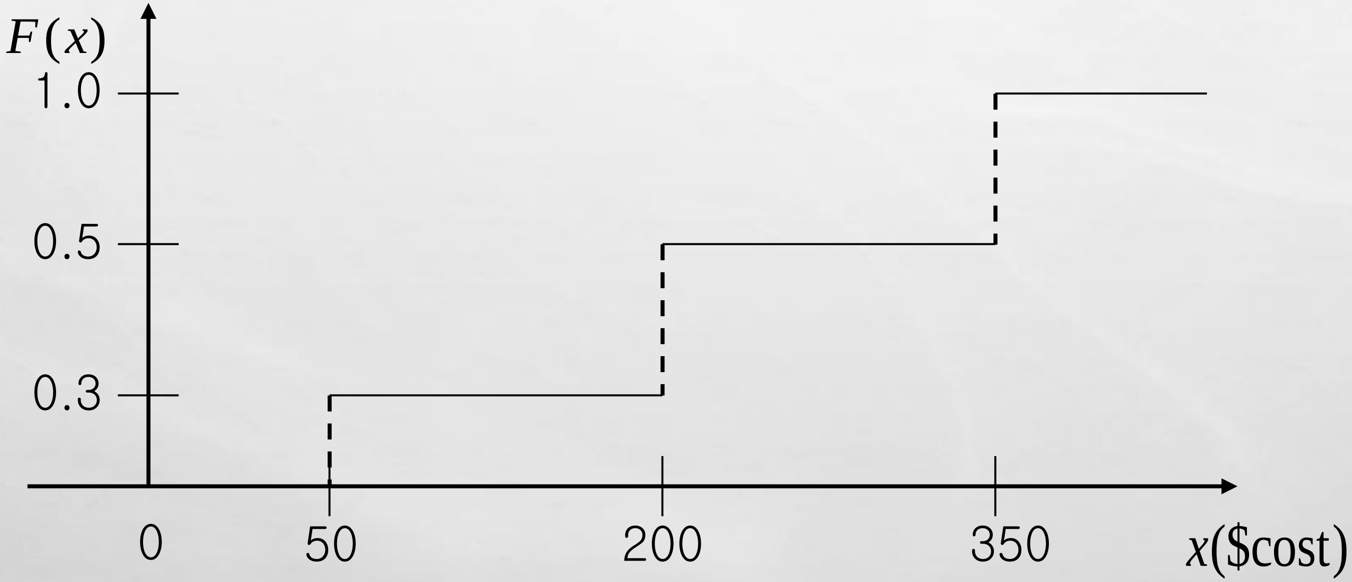
CUMULATIVE DISTRIBUTION FUNCTION

The **cumulative distribution function** of a discrete random variable X is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

CUMULATIVE DISTRIBUTION FUNCTION

- Cumulative Distribution Function
  - Function : $F(x) = P(X \le x)$ $\quad F(x) = \sum_{y:y\le x} P(X = y)$
  - Abbreviation : c. d. f



CUMULATIVE DISTRIBUTION FUNCTION

- Machine Breakdowns

$$-\infty < x < 50 \Rightarrow F(x) = P(\text{cost} \leq x) = 0$$

$$50 \leq x < 200 \Rightarrow F(x) = P(\text{cost} \leq x) = 0.3$$

$$200 \leq x < 350 \Rightarrow F(x) = P(\text{cost} \leq x) = 0.3 + 0.2 = 0.5$$

$$350 \leq x < \infty \Rightarrow F(x) = P(\text{cost} \leq x) = 0.3 + 0.2 + 0.5 = 1.0$$

EXAMPLE

- Roll a coin 3 times, if x= the number of heads and y= the number of tails. Are the x and y independent or not and why? (explain your answer mathematically)

Solution:

S={HHH, HHT, HTH, HTT, THH, THT, TTH,TTT}

R_x={0, 1, 2 , 3}   range of x

R_y={0, 1, 2 , 3}   range of y

P(X=x, Y=y) = P(X=x) P(Y=y)

P(X=0, Y=0) $\neq$ P(X=0) P(Y=0)

0/8 $\neq$ 1/8 . 1/8

x and y are dependent

CONTINUOUS RANDOM VARIABLES

Continuous random variables have an infinite continuum of possible values.

Examples: blood pressure, weight, the speed of a car, the real numbers from 1 to 6.

- **Example of Continuous Random Variables**

- Metal Cylinder Production
  - Suppose that the random variable X is the diameter of a randomly chosen cylinder manufactured by the company.
  - Since this random variable can take **any value between** b and a, it is a **continuous** random variable.

PROBABILITY DENSITY FUNCTION

The **probability density function** (or pdf) $f(x)$ of a continuous random variable is used to determine probabilities from areas as follows:

$$P(a < X < b) = \int_a^b f(x)\, dx \qquad (3\text{-}2)$$

The properties of the pdf are

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x) = 1$

PROBABILITY DENSITY FUNCTION

If X is a continuous random variable, for any x_1 and x_2,

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2)$$

CUMULATIVE DISTRIBUTION FUNCTION

- **Cumulative Distribution Function**
  - $F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)dy$
  - $P(a < X \le b) = P(X \le b) - P(X \le a)$
    $= F(b) - F(a)$
  - $P(a \le X \le b) = P(a < X \le b)$

PROBABILITY DENSITY FUNCTION

Example

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is a positive constant.

a. Find c.
b. Find the CDF of X, $F_X(x)$.
c. Find $P(1 < X < 3)$.

PROBABILITY DENSITY FUNCTION

Solution:-

a. To find c, we can use Property 2 above, in particular

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(u)du \\ &= \int_{0}^{\infty} ce^{-u}du \\ &= c\Big[-e^{-x}\Big]_0^{\infty} \\ &= c. \end{aligned}$$

Thus, we must have $c = 1$.

b. To find the CDF of X, we use $F_X(x) = \int_{-\infty}^{x} f_X(u)du$, so for $x < 0$, we obtain $F_X(x) = 0$. For $x \geq 0$, we have

$$F_X(x) = \int_0^x e^{-u}du = 1 - e^{-x}.$$

Thus,

$$F_X(x) = \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

PROBABILITY DENSITY FUNCTION

c. We can find $P(1 < X < 3)$ using either the CDF or the PDF. If we use the CDF, we have

$$P(1 < X < 3) = F_X(3) - F_X(1) = \left[1 - e^{-3}\right] - \left[1 - e^{-1}\right] = e^{-1} - e^{-3}.$$

Equivalently, we can use the PDF. We have

$$\begin{aligned} P(1 < X < 3) &= \int_1^3 f_X(t)dt \\ &= \int_1^3 e^{-t}dt \\ &= e^{-1} - e^{-3}. \end{aligned}$$