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NAJAF COLLEGE OF TECHNOLOGY
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DIGITAL SIGNAL PROCESSING

3rd YEAR

BY

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DISCRETE FOURIER TRANSFORM

Discrete Fourier Transform

continuous

$$X(F) = \int_{-\infty}^{\infty} x(t)\ e^{-j2\pi F t} dt$$

discrete

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{j2\pi k n}{N}}$$

k-th frequency

evaluating at n of N samples

DISCRETE FOURIER TRANSFORM

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{j2\pi kn}{N}}$$

$\frac{j2\pi kn}{N} \rightarrow b_n$

"kth" frequency bin

$$X_k = x_0 e^{-b_0 j} + x_1 e^{-b_1 j} + \ \dots + x_n e^{-b_{N-1} j}$$

"nth" sample value

Euler's Formula:

$$e^{jx} = cos\,x + j\,sin\,x$$

$$X_k = x_0[\cos(-b_0) + j\sin(-b_0)] + \cdots$$

$$X_k = A_k + B_k j$$

EXAMPLE

Find the 4-Point DFT of the sequence $x(n) = \cos\frac{n\pi}{4}$.

Solution:

Given N =4

$x(n) = \{\cos(0), \cos(\frac{\pi}{4}), \cos(\frac{\pi}{2}), \cos(\frac{3\pi}{4})\} =$ **{1, 0.707, 0, -0.707}**

The N-point DFT of the sequence x(n) is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}, \quad k = 0, 1, 2, \ldots, N-1$$

$$X(k) = \sum_{n=0}^{3} x(n)e^{-j2\pi nk/4}, \quad k = 0, 1, 2, 3$$

$$= \sum_{n=0}^{3} x(n)e^{-j\pi nk/2}, \; k = 0, 1, 2, 3$$

For k = 0

$$X(0) = \sum_{n=0}^{3} x(n)e^{-j\pi(0)n/2} = \sum_{n=0}^{3} x(n) = \mathbf{1}$$

For k = 1

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\pi(1)n/2} = 1 + 0.707e^{-j\pi/2} + 0 + (-0.707)e^{-j3\pi/2}$$

$$= 1 + 0.707(-j) + 0 - (0.707)(j) = \mathbf{1 - j\,1.414}$$

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EXAMPLE

For k = 2

$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi(2)n/2} = 1 + 0.707e^{-j\pi} + 0 + (-0.707)e^{-j3\pi}$

$= 1 + 0.707\,(-1) + 0 - (0.707)(-1) = \mathbf{1}$

For k = 3

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j\pi(3)n/2} = 1 + 0.707e^{-j3\pi/2} + 0 + (-0.707)e^{-j9\pi/2}$$

$= 1 + 0.707\,(j) + 0 - (0.707)(-j) = \mathbf{1 + j\,1.414}$

X(k) $= \{\mathbf{1, 1 - j\,1.414, 1, 1 + j\,1.414}\}$

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EXAMPLE

Find the inverse DFT of $X(k) = \{1, 2, 3, 4\}$.

Solution:

The IDFT is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\,j2\pi nk/N}, \quad n = 0, 1, 2, \dots, N-1$$

Given N = 4, $\quad x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{\,j2\pi nk/4}, \quad n = 0, 1, 2, 3$

When n = 0

$$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{\,j2\pi(0)k/4} = \frac{1}{4}(1 + 2 + 3 + 4) = \mathbf{5/2}$$

When n = 1

$$X(1) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{\,j2\pi(1)k/4} = \frac{1}{4}\left(1 + 2e^{j\pi/2} + 3e^{j\pi} + 4e^{j3\pi/2}\right) =$$

$$\frac{1}{4}(1 + 2(j) + 3(-1) + 4(-j)) = \frac{1}{4}(2 - 2j) = \frac{\mathbf{1}}{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{j}$$

EXAMPLE

When n = 2

$$X(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{\,j2\pi(2)k/4} = \frac{1}{4}\left(1 + 2e^{j\pi} + 3e^{j2\pi} + 4e^{j3\pi}\right) =$$

$$\frac{1}{4}(1 + 2(-1) + 3(1) + 4(-1)) = \frac{1}{4}\,(-2) = -\mathbf{\frac{1}{2}}$$

When n = 3

$$X(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{\,j2\pi(3)k/4} = \frac{1}{4}\left(1 + 2e^{j3\pi/2} + 3e^{j3\pi} + 4e^{j9\pi/2}\right) =$$

$$\frac{1}{4}(1 + 2(-j) + 3(-1) + 4(j)) = -\mathbf{\frac{1}{2}} + \mathbf{\frac{1}{2}}\boldsymbol{j}$$

$$\mathbf{X(n)} = \{\tfrac{5}{2}, \tfrac{1}{2} - \tfrac{1}{2}\,j\,, -\tfrac{1}{2}\,, -\tfrac{1}{2} + \tfrac{1}{2}j\}$$

FAST FOURIER TRANSFORM (FFT) ALGORITHM

- The FFT is a fast algorithm for computing the DFT. FFT is used to increase the computational speed of the DFT. There are two algorithms for FFT:-
- a. Decimation in time algorithm (DIT)
- b. Decimation in frequency algorithm (DIF)
- 1. DIT-FFT
- $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}$, $k = 0, 1, 2, \dots, N-1$
- where $w_N^k = e^{-j2\pi k/N}$, $N = 2,4,8,16,\dots\dots.2^n$

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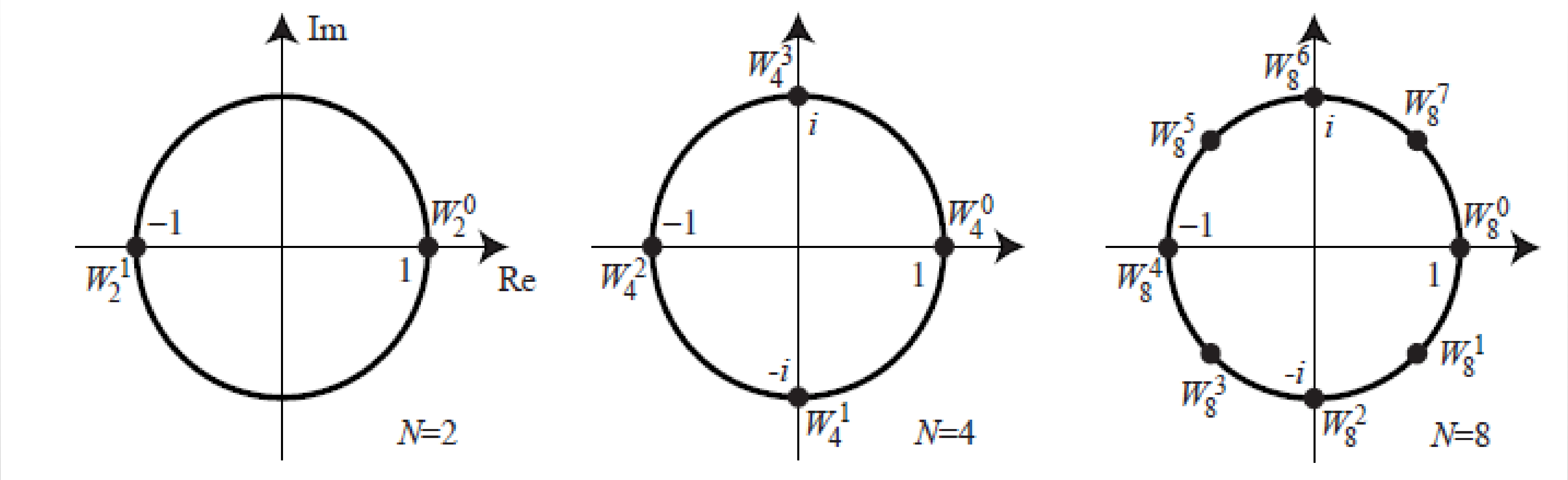
FAST FOURIER TRANSFORM (FFT) ALGORITHM



$W_8^0 = 1+i0$ $\quad$ $W_8^1 = 0.707-i0.707$ $\quad$ $W_8^2 = 0-i=-i$

$W_8^3 = -0.707-i0.707$ $\quad$ $W_8^4 = -1-i0 = -1$ $\quad$ $W_8^5 = -\ 0.707+i0.707$

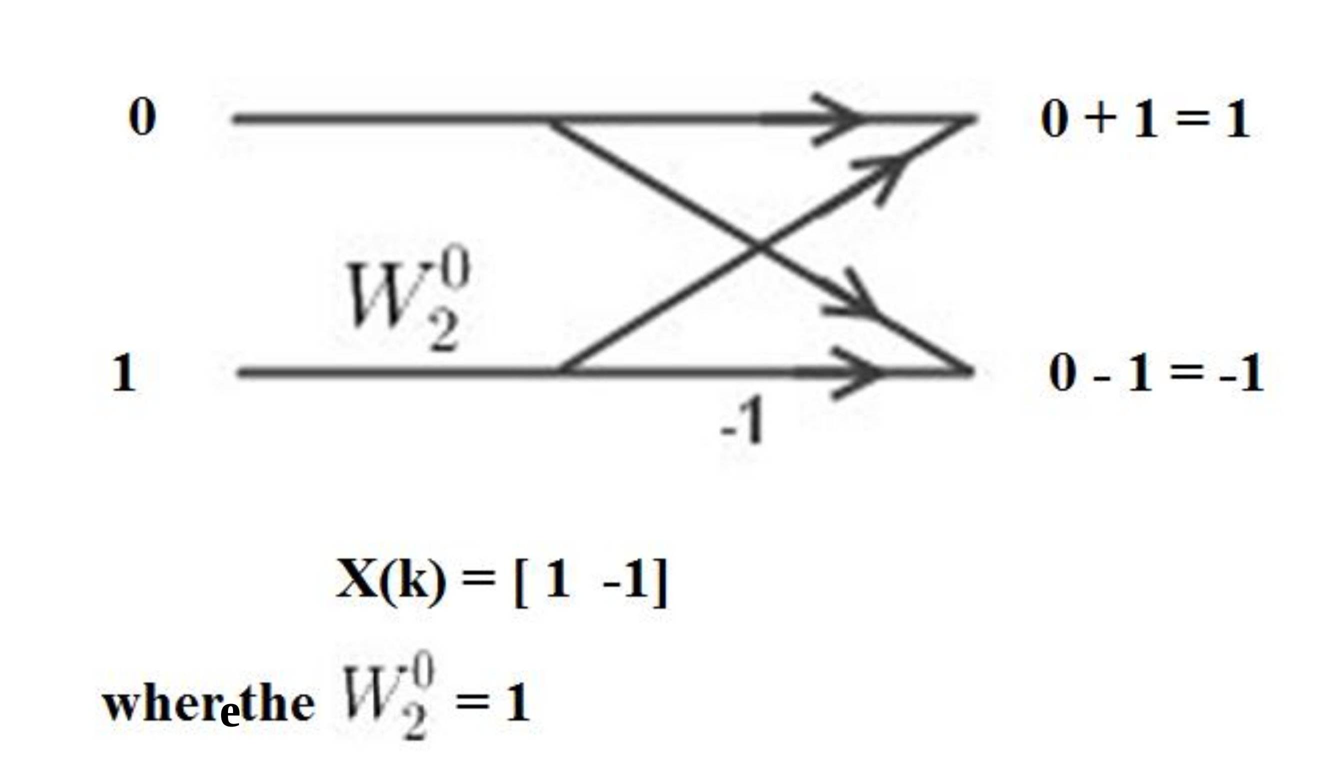
$W_8^6 = 0+i1 = i$ $\quad$ $W_8^7 = 0.707+i0.707$

FLOW-GRAPH FFT

1. **FFT of 2-Samples**

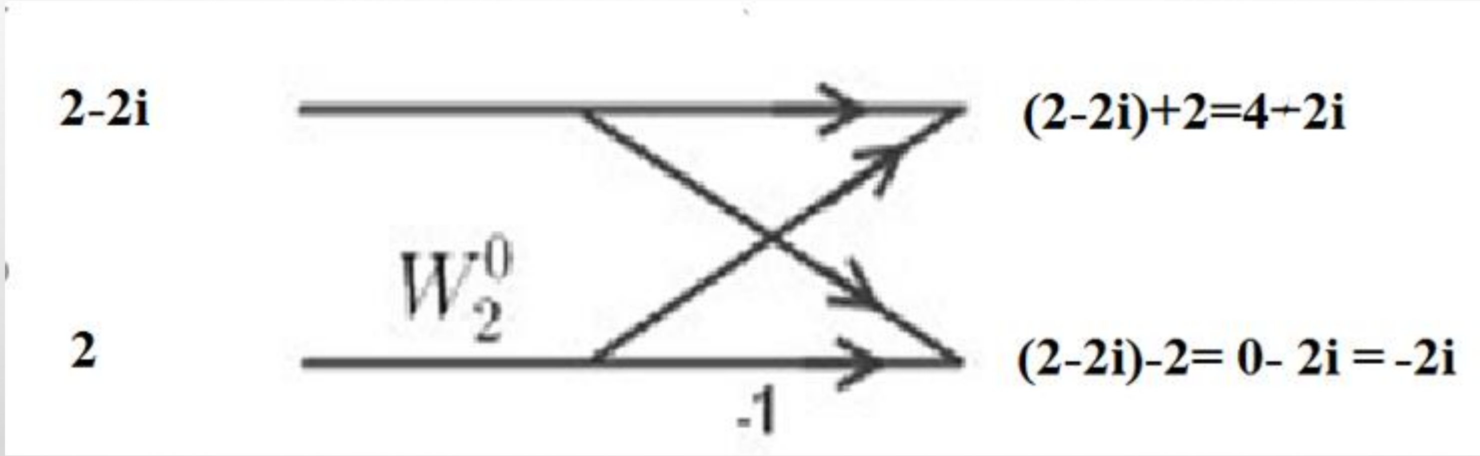
Example: find the FFT for the signal X(n)

X(n)=[0 1]



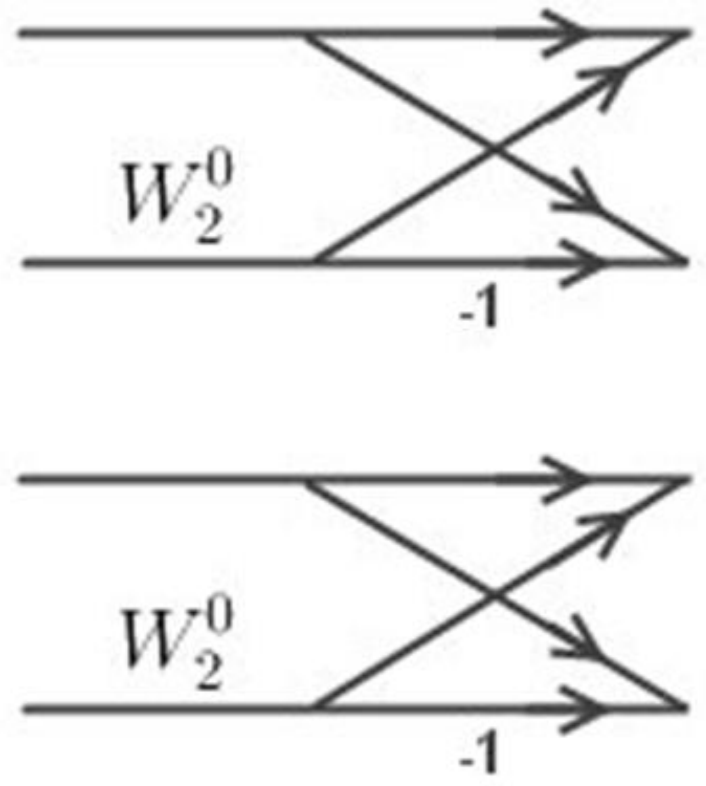
EXAMPLE

Find the FFT for the signal X(n)
X(n)=[2-2i 2]



X(k) = [4-2i   -2i]

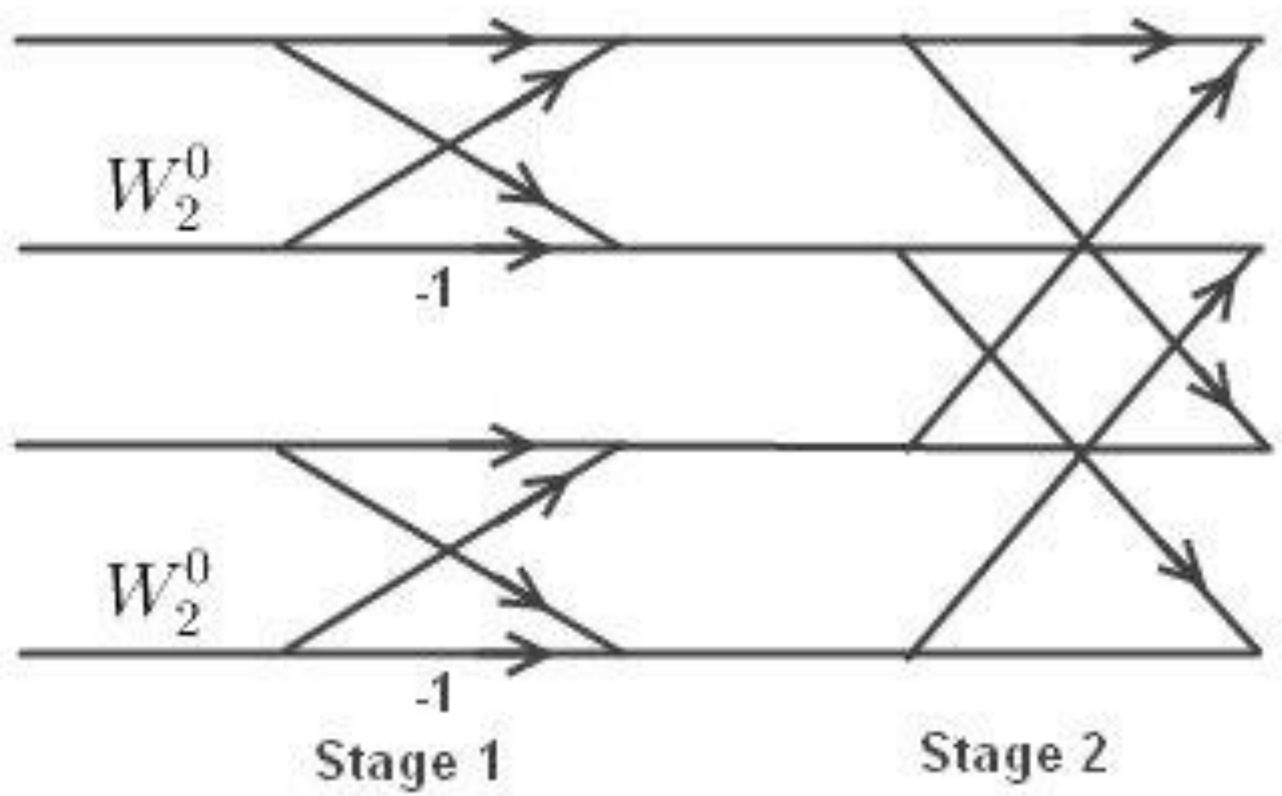
2. FFT OF 4-SAMPLES



Step 1:
Create 2 2-input butterflies.

FFT OF 4-SAMPLES

Step 2: Extend out the lines and then connect the bottom butterfly to the top and the top to the bottom.



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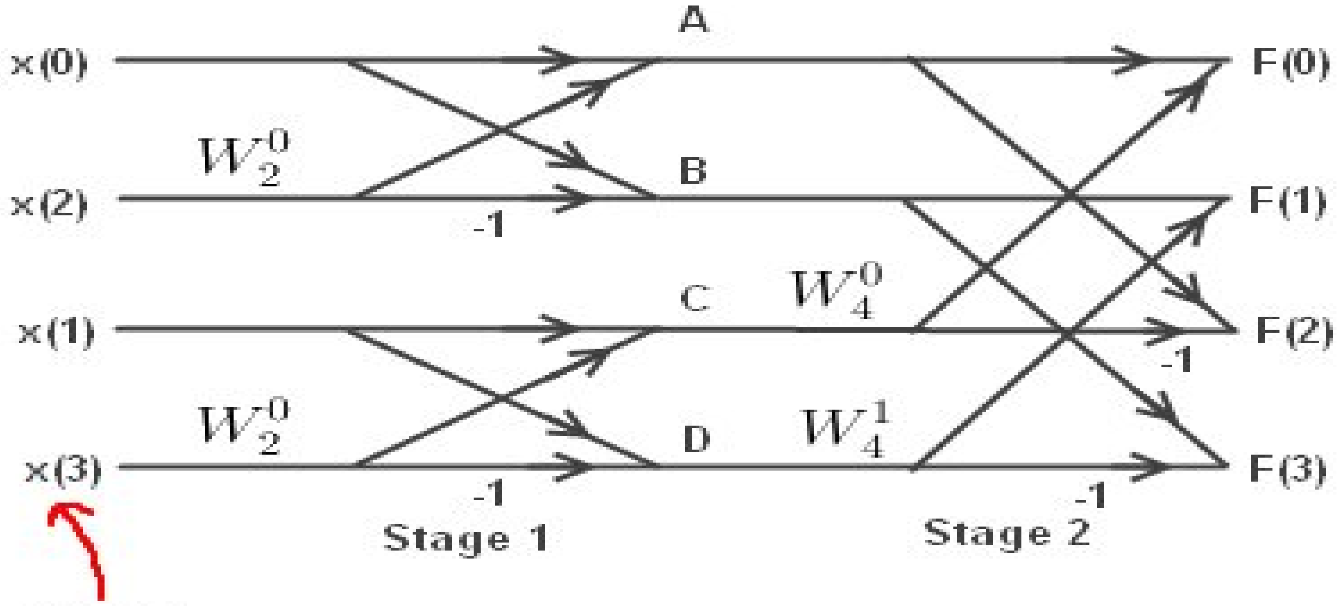
FFT OF 4-SAMPLES

Step 3: Label the input and output values. Label the bottom half of the diagram with W base 4 values, and powers of 0, 1 in order. Note Stage 1 has W base 2, and stage 2 has W base 4. This continues in binary fashion 2, 4, 8, 16 as you add more stages to the butterfly.



Note the reverse bit ordering of input values.

This is the completed 4 input butterfly.

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EXAMPLE

Find the FFT for the signal X(n)

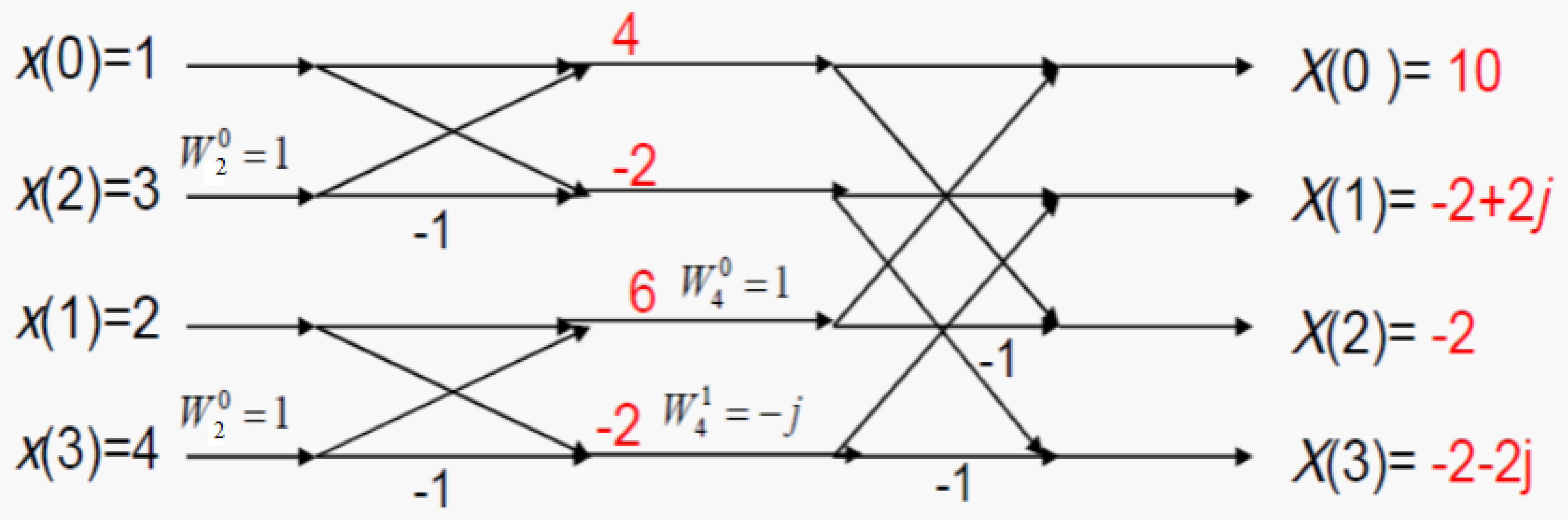
X(n)=[1 2 3 4]

even

odd

$X_e(n)$=[1 3]   **$X_o(n)$=[2 4]**

Solution



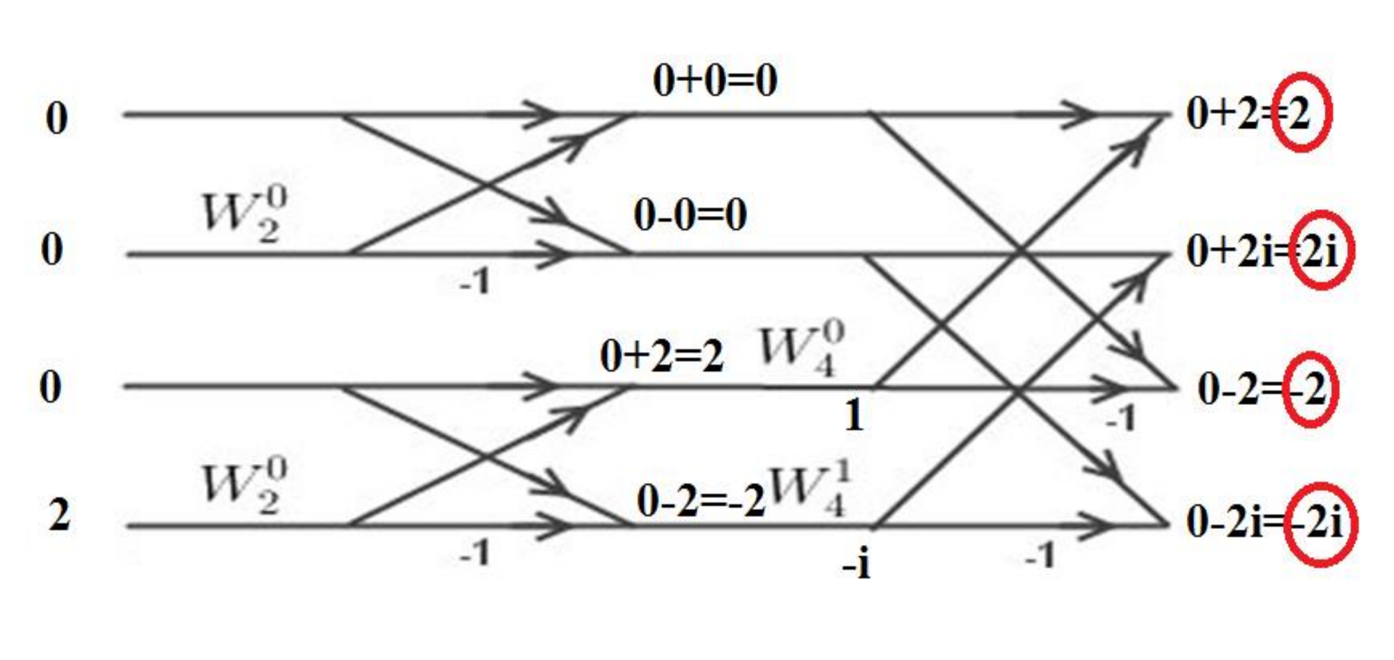
EXAMPLE

Find the FFT for the signal X(n)

X(n)=[0 0 0 2]

X_e(n)=[0 0]

X_o (n)=[0 2]



X(k)=[2 2i -2 -2i]

EXAMPLE

Find the FFT for the signal X(n)

$$x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$$

X(n)=[x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)]

N=8

X_e(n)=[x(0), x(2), x(4), x(6)]   N=4

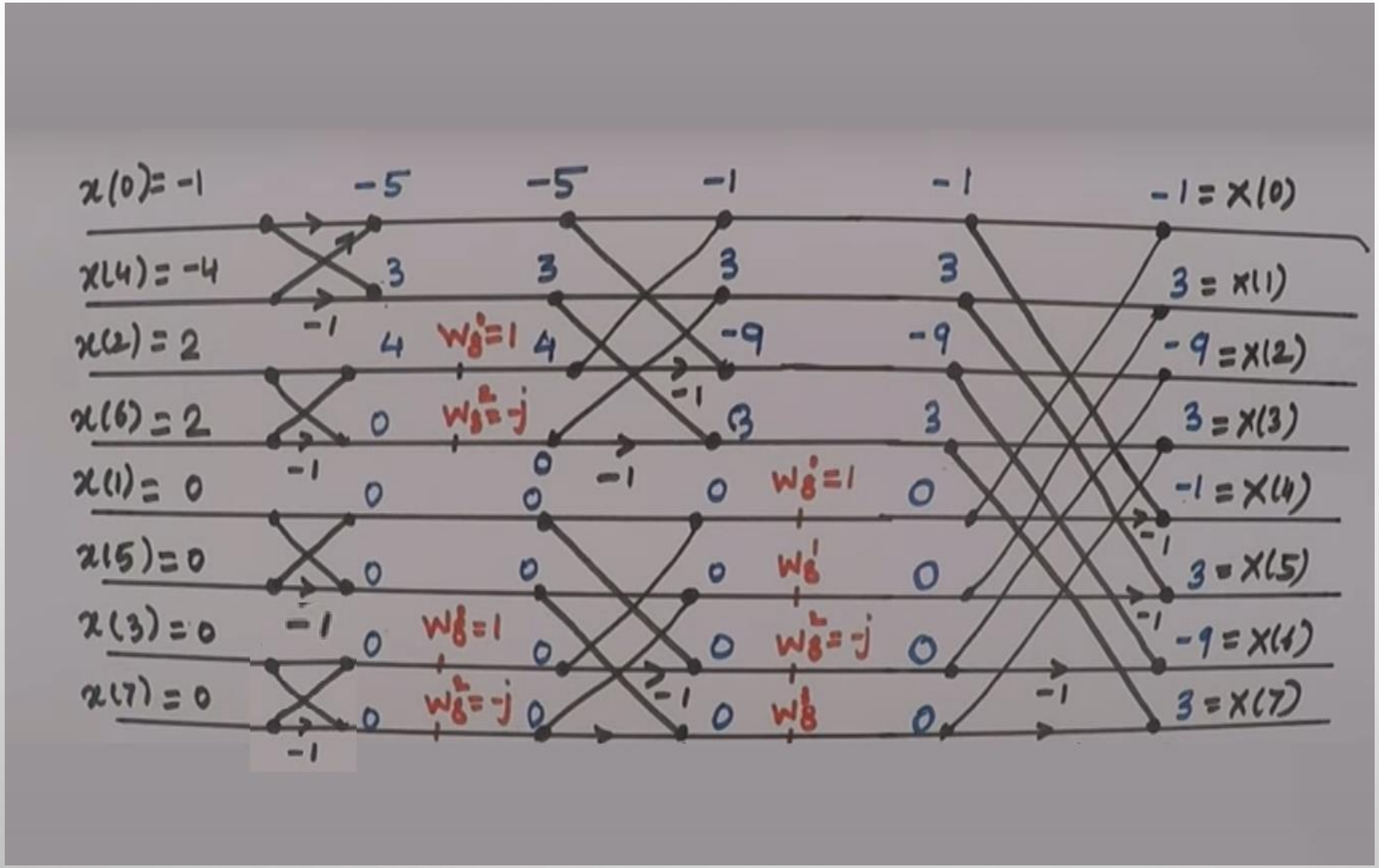
X_o (n)=[x(1), x(3), x(5), x(7)]   N=4

2 pt

$$x_1(n) = \{x(0), x(4)\} \quad x_2(n) = \{x(2), x(6)\}$$

$$x_3(n) = \{x(1), x(5)\} \quad x_4(n) = \{x(3), x(7)\}$$

EXAMPLE



X(k) = [-1 3 -9 3 -1 3 -9 3]

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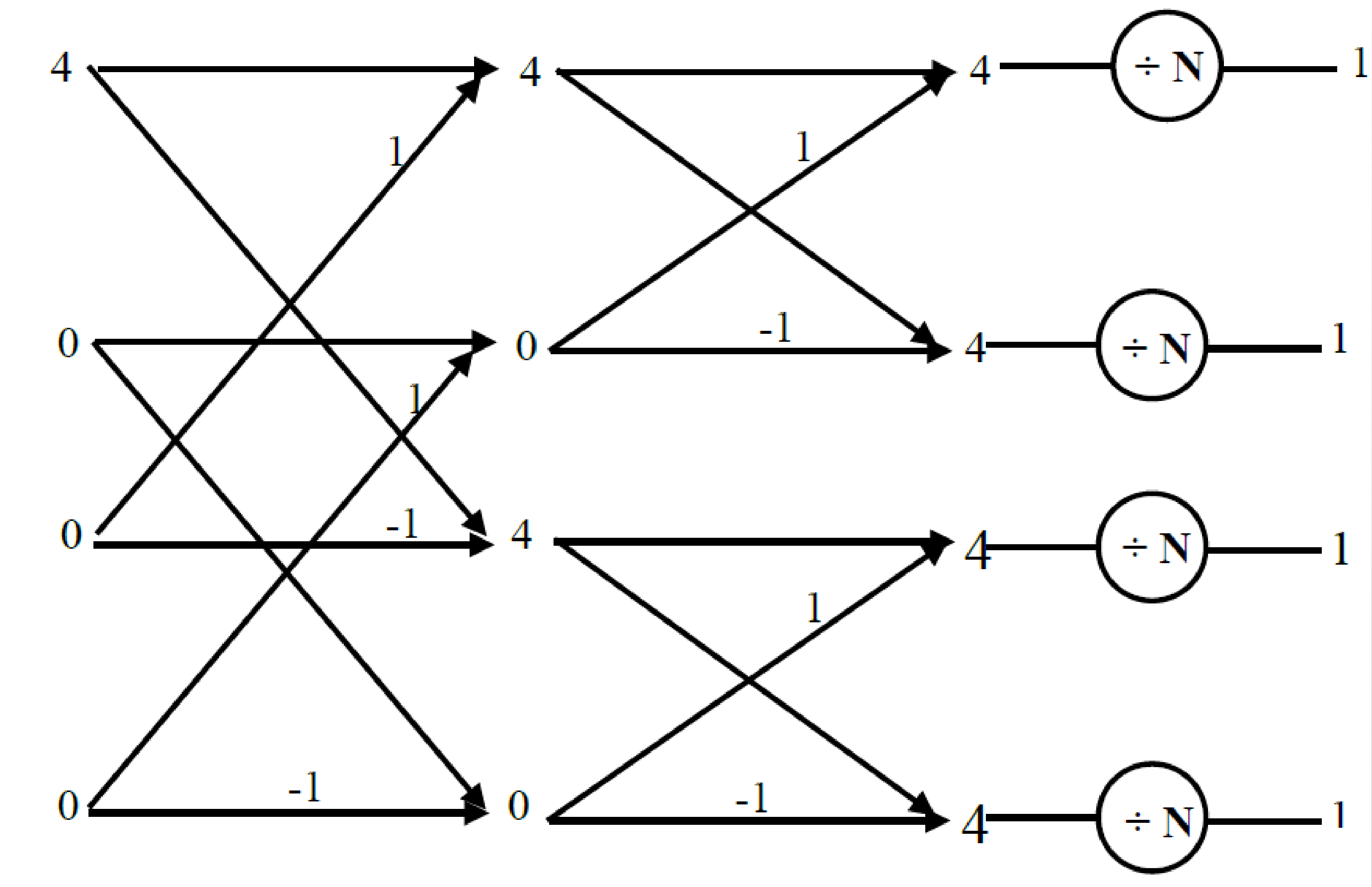
2- DIF- FFT

Example: Find the DIF-FFT for the X[k] = [4 0 0 0].

Solution:

N = 4

x[n] = [1 1 1 1]

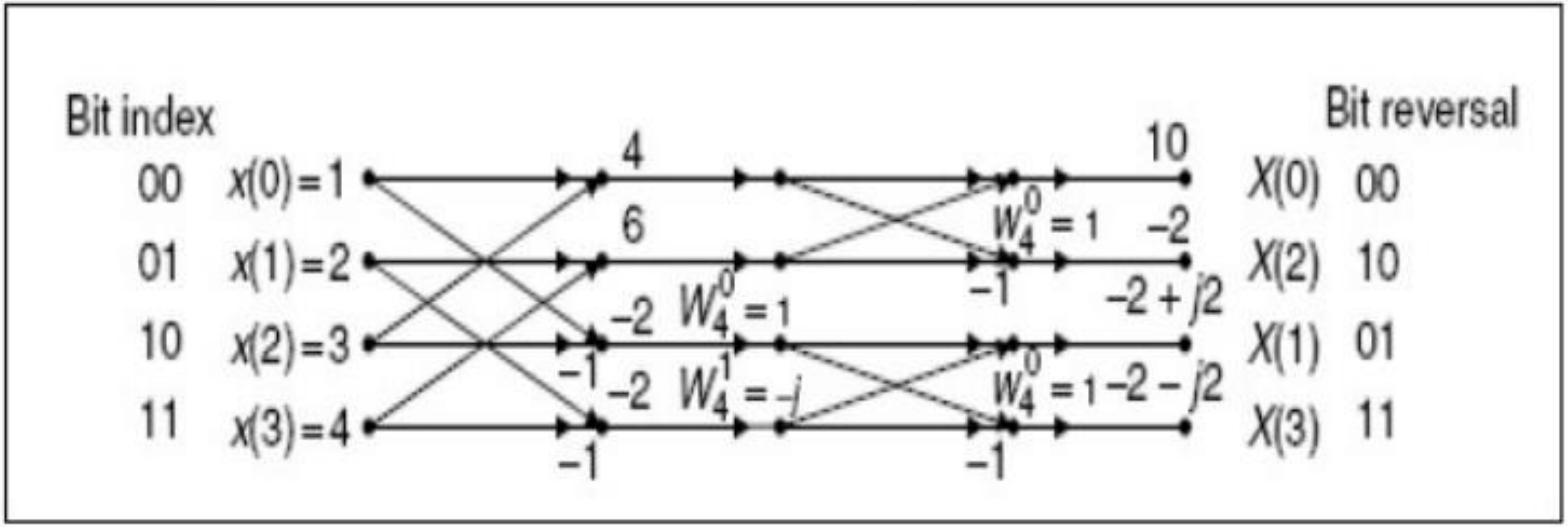


EXAMPLE

- Given a sequence $x(n)$ for $0 \leq n \leq 3$, where $x(0) = 1$, $x(1) = 2$, $x(2) = 3$, and $x(3) = 4$

a. Evaluate its DFT $X(k)$ using the decimation-in-frequency FFT method.

Sol: $W_4^0 = e^{-j\frac{2\pi}{4}(0)} = 1 \; and \; W_4^1 = e^{-j\frac{2\pi}{4}(1)} = -j$



X(n)={10/4=**5/2**, -2/4+2/4j=**-1/2+1/2j**, -2/4=**-1/2**, -2/4-2/4j=**-1/2-1/2j**}

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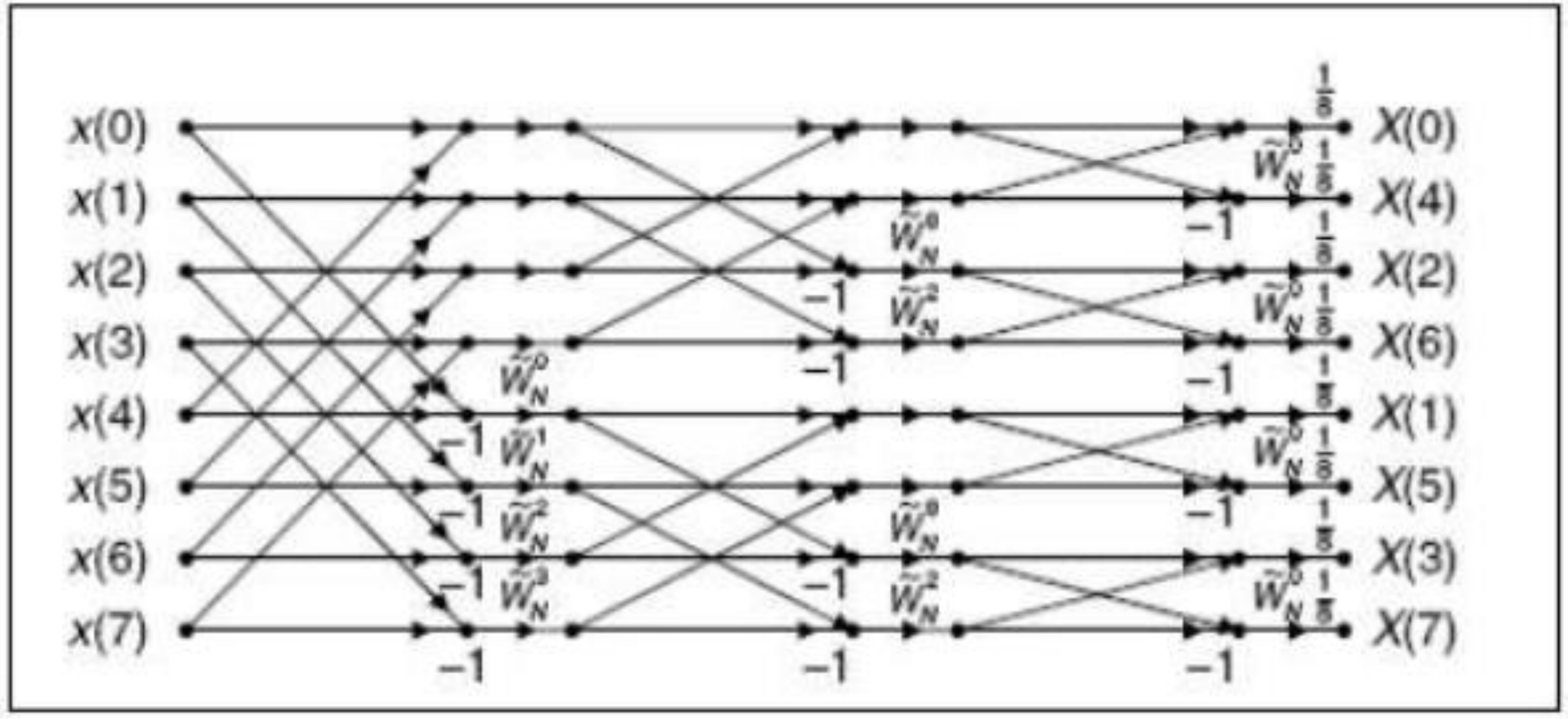
FLOW-GRAPH FFT



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