PROBABILITY, SIGNALS & SYSTEMS

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PROPERTIES OF FOURIER TRANSFORMS

1. Linearity (Superposition) Property
2. Time-Scaling Property
3. Time-Shifting Property
4. Frequency-Shifting Property
5. Time Differentiation Property
6. Frequency Differentiation Property
7. Time Integration Property
8. Time-Frequency Duality Property
9. Convolution Property

LINEARITY (SUPERPOSITION) PROPERTY

Given $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$;

Then $f(t) + g(t) \leftrightarrow F(\omega) + G(\omega)$ **(additivity)**

also $kf(t) \leftrightarrow kF(\omega)$ and $mg(t) \leftrightarrow mG(\omega)$ **(homogeneity)**

Note: k and m are constants

Combining these we have,

$$kf(t) + mg(t) \leftrightarrow kF(\omega) + mG(\omega)$$

Hence, the Fourier Transform is a linear transformation.

TIME SCALING PROPERTY

$$\mathfrak{F}\{f(at)\} = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

$$\mathfrak{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)e^{-j\omega t}dt$$

Let $\lambda = at$ & $d\lambda = adt,$

$$\mathfrak{F}\{f(at)\} = \int_{-\infty}^{\infty} f(\lambda)e^{-j\omega t}\frac{d\lambda}{a} = \frac{1}{a}F\left(\frac{\omega}{a}\right)$$

Hence, $\mathfrak{F}\{f(-t)\} = F(-\omega) = F^{*}(\omega)$

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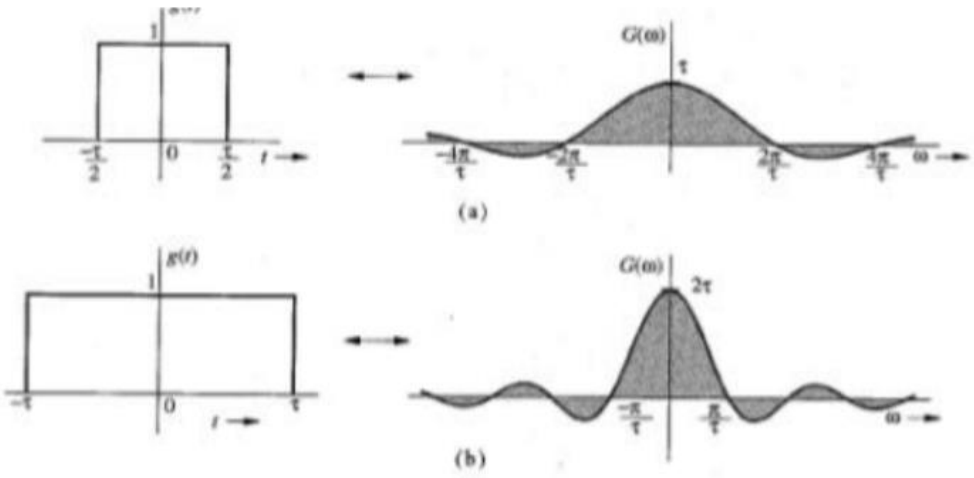
TIME SCALING PROPERTY

$$\mathfrak{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Time compression of a signal results in spectral expansion and time expansion of a signal results in spectral compression.



TIME SHIFTING PROPERTY

$$\mathfrak{F}\left\{f(t-t_0)\right\} = e^{-j\omega t_0} F\left(\omega\right)$$

$$\mathfrak{F}\left\{f(t-t_0)\right\} = \int_{-\infty}^{\infty} f(t-t_0)e^{-j\omega t}dt$$

Let $\lambda = t - t_0,\ d\lambda = dt\ \&\ t = \lambda + t_0$

$$\mathfrak{F}\left\{f(t-t_0)\right\} = \int_{-\infty}^{\infty} f(\lambda)e^{-j\omega(\lambda+t_0)}d\lambda =$$

$$= e^{-j\omega t_9}\int_{-\infty}^{\infty} f(\lambda)e^{-j\omega\lambda}d\lambda = e^{-j\omega t_0}F\left(\omega\right)$$

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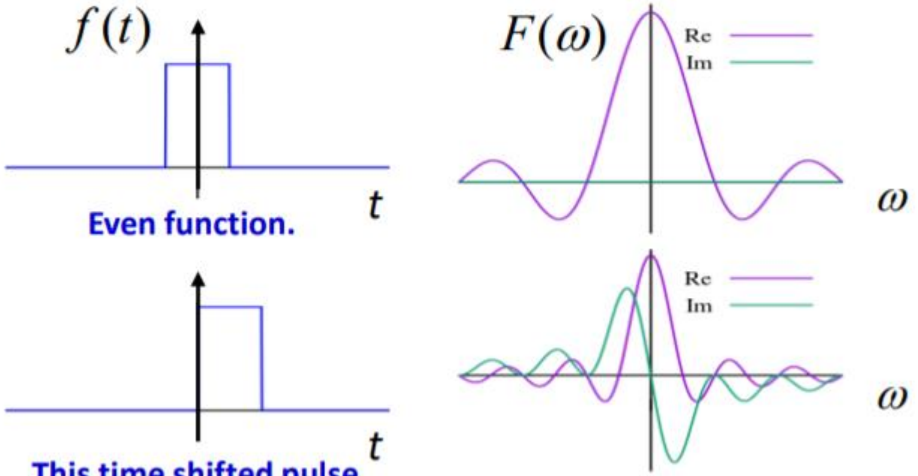
TIME SHIFTING PROPERTY

$$\mathfrak{F}\left\{f(t-t_0)\right\} = e^{-j\omega t_0} F(\omega)$$

Delaying a signal by t_0 seconds does not change its amplitude spectrum, but the phase spectrum is changed by $-2\pi f t_0$.
Note that the phase spectrum shift changes linearly with frequency f.

$$|F(\omega)| = \sqrt{\left[\mathrm{Re}\left(F(\omega)\right)\right]^2 + \left[\mathrm{Im}\left(F(\omega)\right)\right]^2}$$

A time shift produces a phase shift in its spectrum.



This time shifted pulse is both even and odd.

Both must be identical.

FREQUENCY SHIFTING PROPERTY

$$\mathfrak{F}\left\{ f(t)e^{j\omega_0 t} \right\} = F\left(\omega - \omega_0\right)$$

$$\mathfrak{F} f(t)e^{j\omega_0 t}\right\} = \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}d\,\mathbf{t} = F\left(\omega - \omega_0\right)$$

Special application:

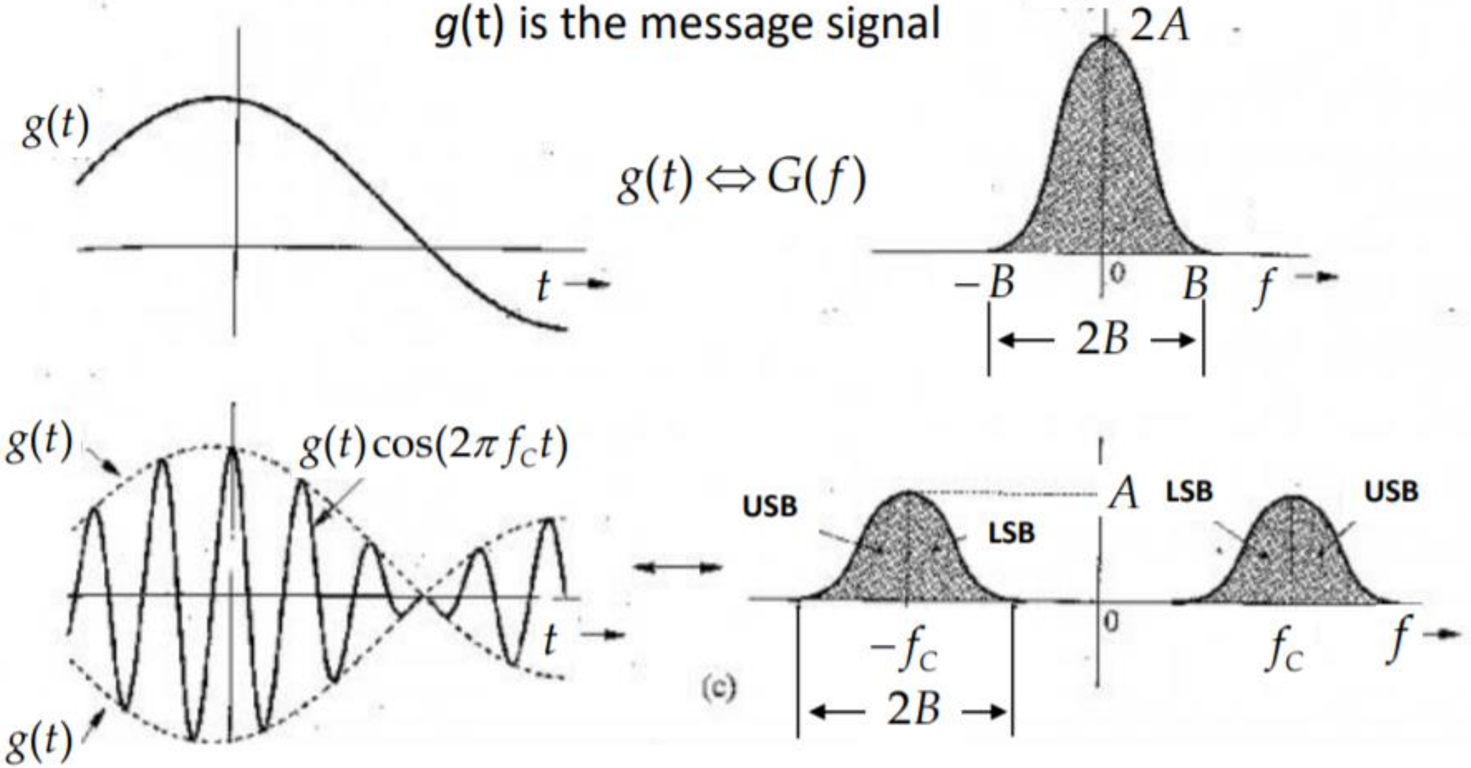
Apply to $\cos\left(\omega_0 t\right) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right);$

$$\mathfrak{F}\left\{ f(t)\cos\left(\omega_0 t\right)\right\} = \frac{1}{2}\left(F(\omega - \omega_0) + F(\omega + \omega_0)\right)$$

FREQUENCY SHIFTING PROPERTY

Multiplication of a signal $g(t)$ by the factor $[\cos(2\pi f_C t)]$ places $G(f)$ centered at $f = \pm f_C$.

Carrier frequency is f_c & g(t) is the message signal



AN IMPORTANT FORMULA TO REMEMBER

Euler's formula

$$\exp[\pm j\theta] = \cos(\theta) \pm j\sin(\theta)$$

$$\sin(\theta) = \frac{1}{2j}\left(\exp[j\theta] - \exp[-j\theta]\right)$$

$$\cos(\theta) = \frac{1}{2}\left(\exp[j\theta] + \exp[-j\theta]\right)$$

$$e^{\pm j(\pi/2)} = \pm j \quad and \quad e^{\pm jn\pi} = \begin{cases} 1 & \text{for } n \text{ even} \\ -1 & \text{for } n \text{ odd} \end{cases}$$

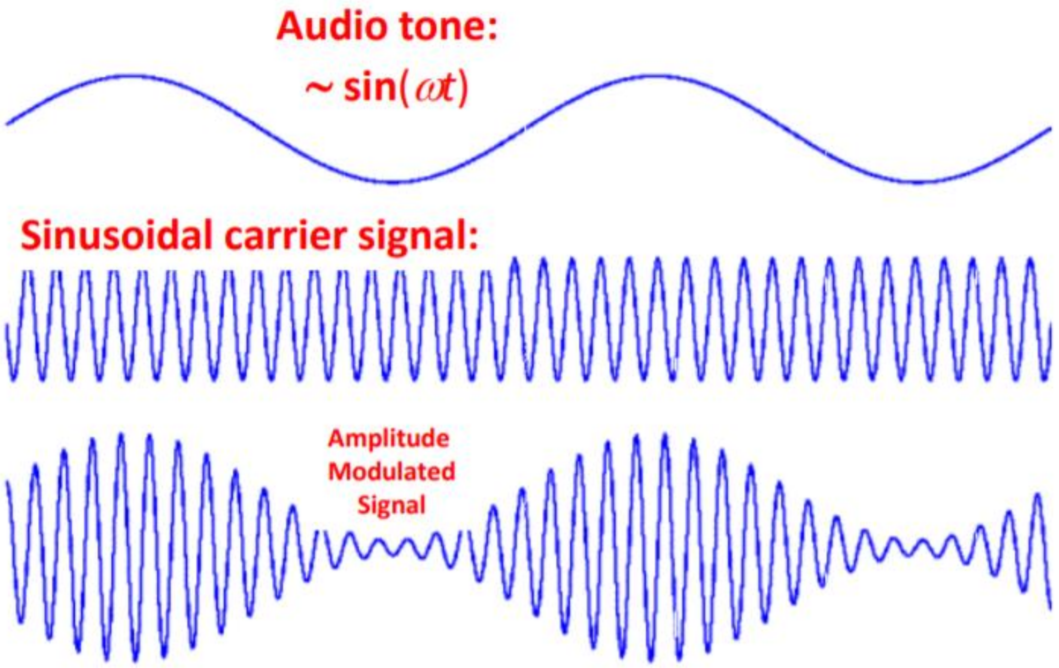
$$a + jb = re^{j\theta} \quad where \quad r = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

MODULATION COMES FROM FREQUENCY SHIFTING PROPERTY

Given FT pair: $f(t) \Leftrightarrow F(\omega)$

then, $f(t)e^{j\omega_0 t} \Leftrightarrow F(\omega - \omega_0)$

Amplitude Modulation Example:



TRANSFORM DUALITY PROPERTY

$$g(t) \Leftrightarrow G(f)$$

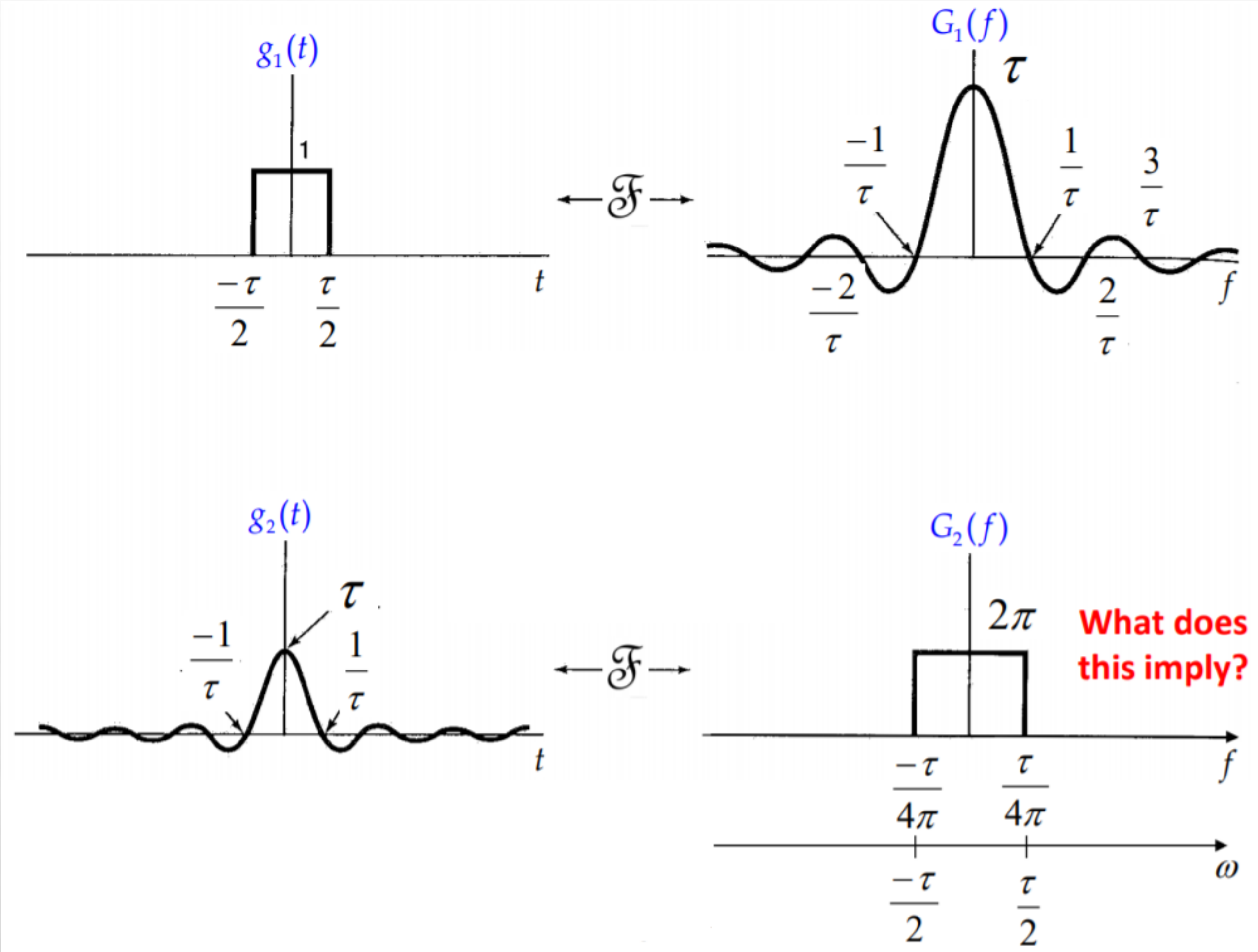
and

$$G(t) \Leftrightarrow g(-f)$$

Note the minus sign!

Because of the minus sign they are not perfectly symmetrical – See the illustration on next slide.

TRANSFORM DUALITY PROPERTY



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PARSEVAL'S THEOREM

$$E \equiv \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} f(t) f^*(t) \, dt$$

If f(t) is real function

$$E \equiv \int_{-\infty}^{\infty} f^2(t) \, dt = \int_{-\infty}^{\infty} f(t) f^*(t) \, dt$$

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$$

$$\int_{-\infty}^{\infty} f(t) \, [\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) e^{-j\omega t} \, d\omega] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt] F^*(j\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \, F^*(j\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega$$

EXAMPLE 1

$$e(t) = \begin{cases} e^{-at}, & a > 0, t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$F\{e(t)\} = \int_{-\infty}^{\infty} e^{-at}\, e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \frac{1}{-(a+j\omega)} e^{-(a+j\omega)t} \Big|_{t=0}^{t=\infty}$$

$$= \frac{e^{-\infty}}{-(a+j\omega)} - \frac{e^{-0}}{-(a+j\omega)}$$

$$= \frac{1}{(a+j\omega)}, a > 0$$

EXAMPLE 2

$$e(t) = \begin{cases} e^{at}, & t < 0 \\ e^{-at}, & t \geq 0 \end{cases}$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{at}\, e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t} dt$$

$$= \frac{1}{(a - j\omega)} + \frac{1}{(a + j\omega)} = \frac{2a}{a^2 + \omega^2}$$

EXAMPLE 3

$x(t) = u(t)e^{-at}\, sin(w_0 t)$

We can write it as:

$\frac{u(t)}{2j}(e^{jw_0 t} - e^{-jw_0 t})\, e^{-at}$

$\frac{u(t)}{2j}(e^{-(a-jw_0)t} - e^{-(a+jw_0)t})$

$u(t)\, e^{-at} \longleftrightarrow \frac{1}{(a+j\omega)}$

$u(t)\, e^{-(a-jw_0)t} \longleftrightarrow \frac{1}{(a-jw_0+j\omega)}$

And

$u(t)\, e^{-(a+jw_0)t} \longleftrightarrow \frac{1}{(a+jw_0+j\omega)}$

$\frac{u(t)}{2j}(e^{-(a-jw_0)t} - e^{-(a+jw_0)t}) \longleftrightarrow \frac{1}{2j}\left[\frac{1}{(a-jw_0+j\omega)} - \frac{1}{(a+jw_0+j\omega)}\right]$

$u(t)e^{-at}\sin(w_0 t) \longleftrightarrow \frac{w_0}{a^2+w_0{}^2-w^2+j2aw}$

$= \frac{w_0}{(a+jw)^2-w_0{}^2}$