

**AL FURAT AL AWSAT TECHNICAL UNIVERSITY  
NAJAF COLLEGE OF TECHNOLOGY  
DEPARTMENT OF AVIONICS ENGINEERING**

**DIGITAL SIGNAL PROCESSING  
3<sup>rd</sup> YEAR**

**BY  
RUAA SHALLAL ANOOZ**

# LAPLACE TRANSFORM

- The transform of choice for analog filters and continuous-time systems in general is Laplace transform

$$F(s) = L\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt$$

where  $s$  is a complex variable of the form  $s = \sigma + j\omega$ . Signal  $x(t)$  can be recovered from  $X(s)$  by applying the inverse Laplace transform which is given by

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st} ds$$

# APPLICATIONS OF LAPLACE TRANSFORM

- Applications of Laplace Transforms/
  - Circuit analysis
    - Easier than solving differential equations
    - Provides the general solution to any arbitrary wave (not just LRC)
      - Transient
      - Sinusoidal steady-state-response (Phasors)
  - Signal processing
  - Communications

# EXAMPLES

**Ex1:** let  $f(t) = 1$  find  $F(S)$ ?

**Sol:**

$$F(s) = \int_0^{\infty} e^{-st} dt = -\frac{1}{s}(0 - 1) = \frac{1}{s}$$

**Ex2:** let  $f(t) = e^{-at}$  find  $F(S)$ ?

**Sol:**

$$F(s) = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt = \frac{1}{s+a}$$

# EXAMPLES

**Ex3:** let  $f(t) = \sin t$  find  $F(S)$ ?

**Sol:**

$$F(s) = \int_0^{\infty} e^{-st} \sin(t) dt \quad \leftarrow \text{Integrate by parts}$$

$$\therefore \int_0^{\infty} e^{-st} \sin(t) dt = [-e^{-st} \cos(t)]_0^{\infty} - s \int_0^{\infty} e^{-st} \cos(t) dt =$$

$$-e^{-st}(1) - s \int_0^{\infty} e^{-st} \cos(t) dt$$

**let**  $u = e^{-st}, du = -se^{-st} dt$   
 $dv = \cos(t) dt, v = \sin(t)$

$$\therefore \int_0^{\infty} e^{-st} \cos(t) dt =$$

$$[-e^{-st} \sin(t)]_0^{\infty} + s \int_0^{\infty} e^{-st} \sin(t) dt = -e^{-st}(0) + s \int_0^{\infty} e^{-st} \sin(t) dt$$

**REMEMBER**  $\int u dv = uv - \int v du$

**let**  $u = e^{-st}, du = -se^{-st} dt$   
 $dv = \sin(t) dt, v = -\cos(t)$

**Substituting, we get:**

$$\int_0^{\infty} se^{-st} \sin(t) dt = 1 + s^2 \int_0^{\infty} e^{-st} \sin(t) dt =$$

$$(1 + s^2) \int_0^{\infty} e^{-st} \sin(t) dt = 1$$

$$\int_0^{\infty} e^{-st} \sin(t) dt = \frac{1}{1 + s^2}$$

# LAPLACE TRANSFORM PROPERTIES

Addition/Scaling

$$L[af_1(t) \pm bf_2(t)] = aF_1(s) \pm bF_2(s)$$

Differentiation

$$L\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$$

Integration

$$L\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)dt\right]_{t=0\pm}$$

Convolution

$$\int_0^t f_1(t-\tau)f_2(\tau)d\tau = F_1(s)F_2(s)$$

Initial – value theorem

$$f(0+) = \lim_{s \rightarrow \infty} sF(s)$$

Final – value theorem

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Scaling in Time

$$L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Time Shift

$$L\{f(t-t_0)u(t-t_0)\} = e^{-st_0}F(s)$$

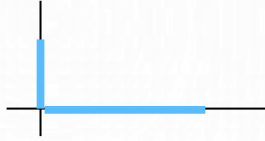

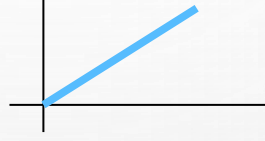
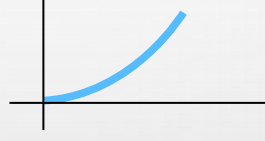
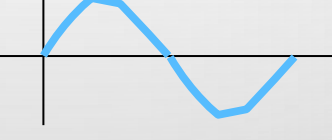
S-plane (frequency) shift

$$L\{e^{-at}f(t)\} = F(s+a)$$

Multiplication by  $t^n$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

# TABLE OF SELECTED LAPLACE TRANSFORMS

Name	$f(t)$		$F(s)$
Impulse	$f(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$		1
Step	$f(t) = 1$		$\frac{1}{s}$
Ramp	$f(t) = t$		$\frac{1}{s^2}$
Exponential	$f(t) = e^{at}$		$\frac{1}{s - a}$
Sine	$f(t) = \sin(\omega t)$		$\frac{1}{\omega^2 + s^2}$

# MORE TRANSFORMS

$$f(t) = t^n u(t) \Leftrightarrow F(s) = \frac{n!}{s^{n+1}}$$

$$n = 0, f(t) = u(t) \Leftrightarrow F(s) = \frac{0!}{s^1} = \frac{1}{s}$$

$$n = 1, f(t) = tu(t) \Leftrightarrow F(s) = \frac{1!}{s^2}$$

$$n = 5, f(t) = t^5 u(t) \Leftrightarrow F(s) = \frac{5!}{s^6} = \frac{120}{s^6}$$

$$f(t) = \delta(t) \Leftrightarrow F(s) = 1$$



# EXAMPLES

**Ex:** Find  $L\{\frac{1}{2}e^t - \frac{1}{2}e^{-t}\}$

**Sol:**

$$\begin{aligned} &= \frac{1}{2}L\{e^t\} - \frac{1}{2}L\{e^{-t}\} \\ &= \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) \\ &= \frac{1}{2}\left(\frac{(s+1) - (s-1)}{s^2 - 1}\right) \\ &= \frac{1}{s^2 - 1} \end{aligned}$$

# EXAMPLES

**Ex:** Find  $L\{d/dt \cos(t)\}$

**Sol:**

$$= \frac{s^2}{s^2+1} - f(0^+)$$

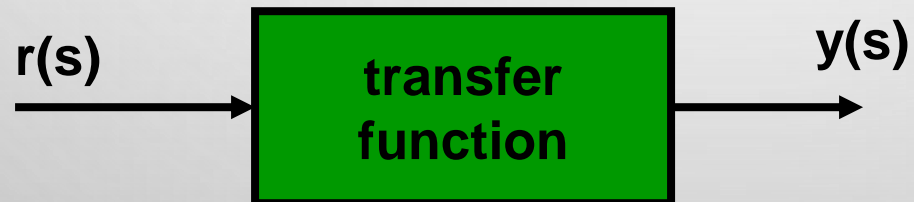
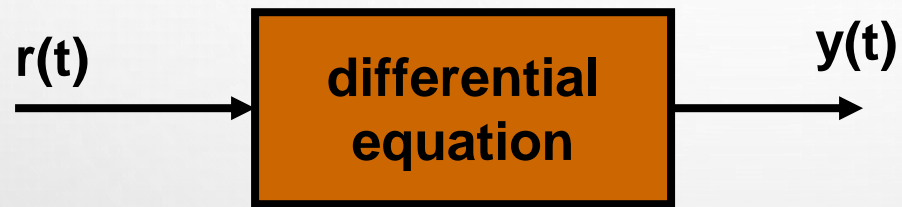
$$= \frac{s^2}{s^2+1} - 1$$

$$= \frac{s^2 - (s^2 + 1)}{s^2 + 1}$$

$$\frac{-1}{s^2 + 1} = L\{-\sin(t)\}$$

# TRANSFER FUNCTIONS

- Definition : a transfer function is an expression that relates the output to the input in the s-domain

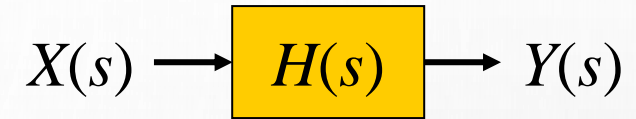


# TRANSFER FUNCTIONS

- Definition

- $H(s) = Y(s) / X(s)$

- Relates the output of a linear system (or component) to its input
- Describes how a linear system responds to an impulse
- All linear operations allowed
  - Scaling, addition, multiplication

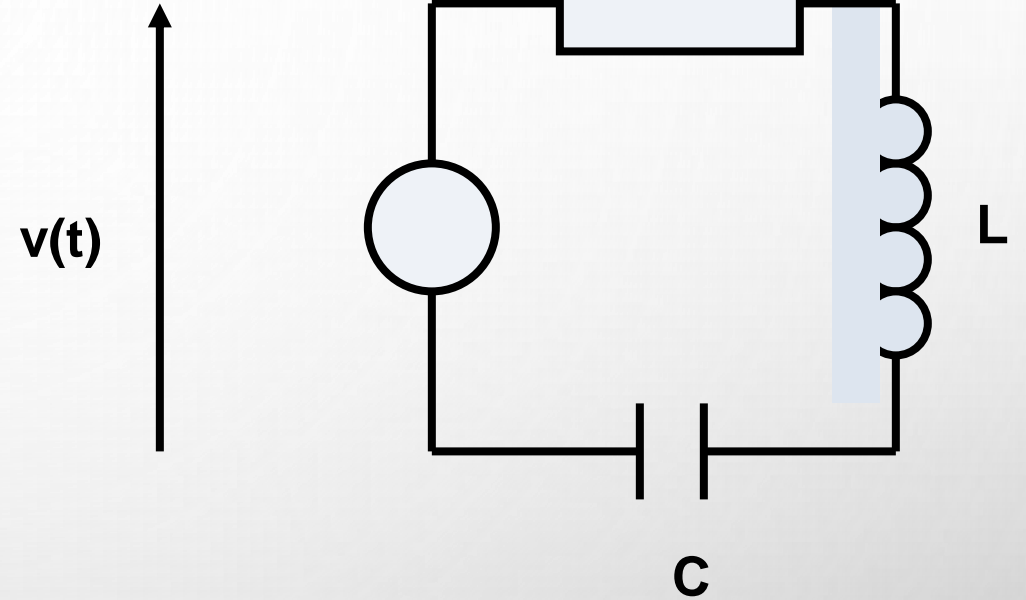


# EXAMPLE

$$v(t) = R I(t) + 1/C \int I(t) dt + L di(t)/dt$$

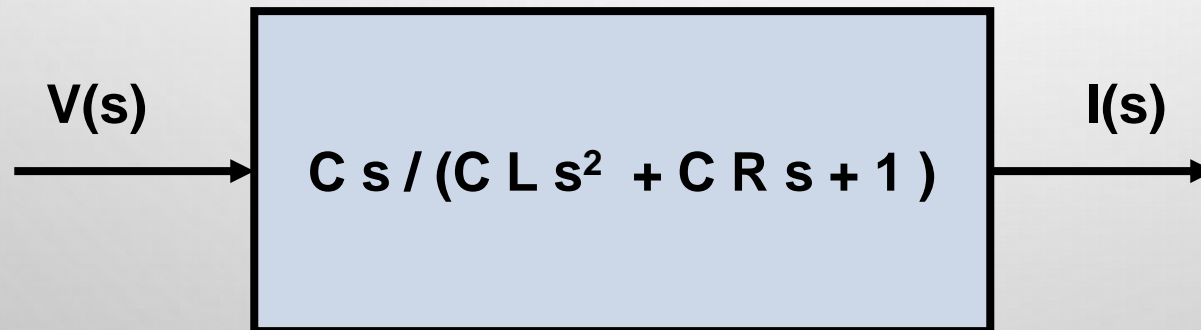
$$V(s) = [R I(s) + 1/(C s) I(s) + s L I(s)]$$

**Note: Ignore initial conditions**



## EXAMPLE

- $V(s) = (R + 1/(C s) + s L) I(s)$   
 $= (C L s^2 + C R s + 1) / (C s) I(s)$
- $I(s)/V(s) = C s / (C L s^2 + C R s + 1)$



# EXAMPLE

**Ex:** Consider an RL circuit with  $R=4$ ,  $L=1/2$ . Find  $i(t)$  if  $v(t)=12u(t)$ .

$$L \frac{di(t)}{dt} + Ri(t) = v(t)$$

$$0.5 \frac{di(t)}{dt} + 4i(t) = v(t)$$

Using Laplace:

$$(0.5s + 4)I(s) = V(s)$$

$$H(s) = I(s)/V(s) = \frac{1}{0.5s + 4} \Rightarrow I(s) = H(s).V(s)$$

$$v(t) = 12u(t) \leftrightarrow V(s) = 12/s$$

$$I(s) = H(s).V(s) = \frac{24}{s(s+8)} = \frac{k_1}{s} + \frac{k_2}{s+8}$$

$$k_1 = [(s - p_1)I(s)] \Big|_{p_1=0} = 3$$

$$k_2 = [(s - p_2)I(s)] \Big|_{p_2=-8} = -3$$

$$\Rightarrow I(s) = \frac{3}{s} + \frac{-3}{s+8} \Leftrightarrow i(t) = 3u(t) - 3e^{-8t}; t > 0$$

