AL FURAT AL AWSAT TECHNICAL UNIVERSITY
NAJAF COLLEGE OF TECHNOLOGY
DEPARTMENT OF AVIONICS ENGINEERING

DIGITAL SIGNAL PROCESSING

3rd YEAR

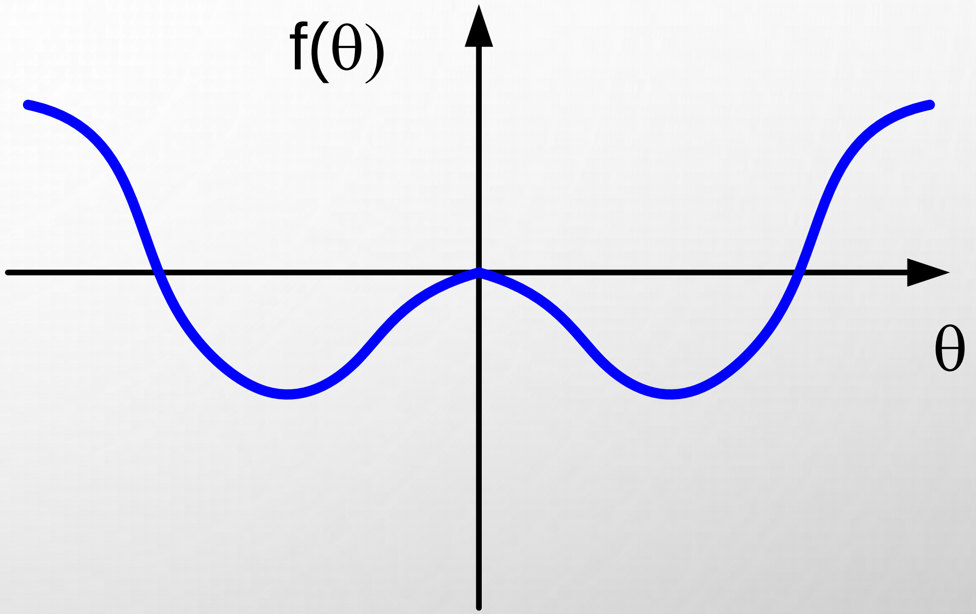
BY

RUAA SHALLAL ANOOZ

Fourier series

1. Even Functions

The value of the function would be the same when we walk equal distances along the X-axis in opposite directions.



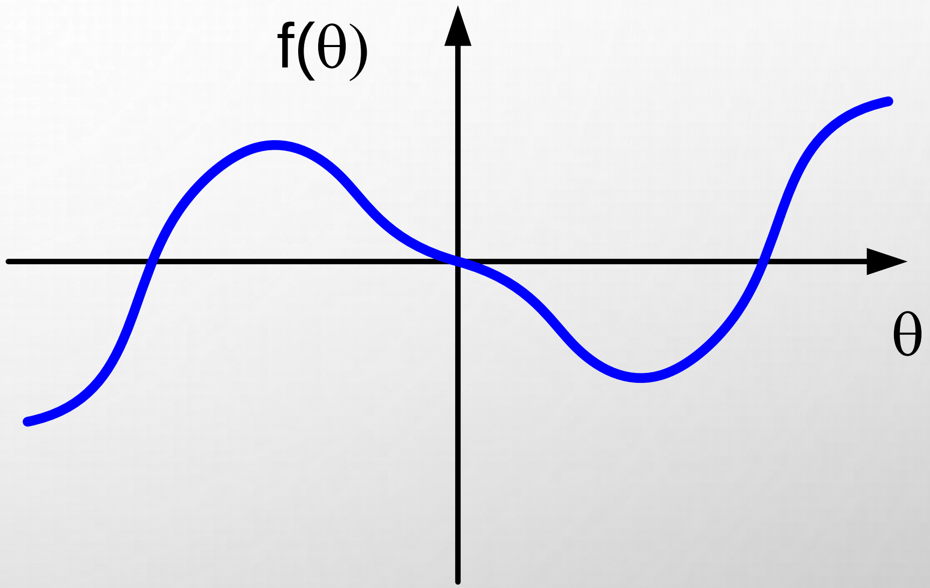
Mathematically speaking :

$f(-\theta) = f(\theta)$

Fourier series

2. Odd Functions

The value of the function would change its sign but with the same magnitude when we walk equal distances along the X-axis in opposite directions.



Mathematically speaking :

$f(-\theta) = -f(\theta)$

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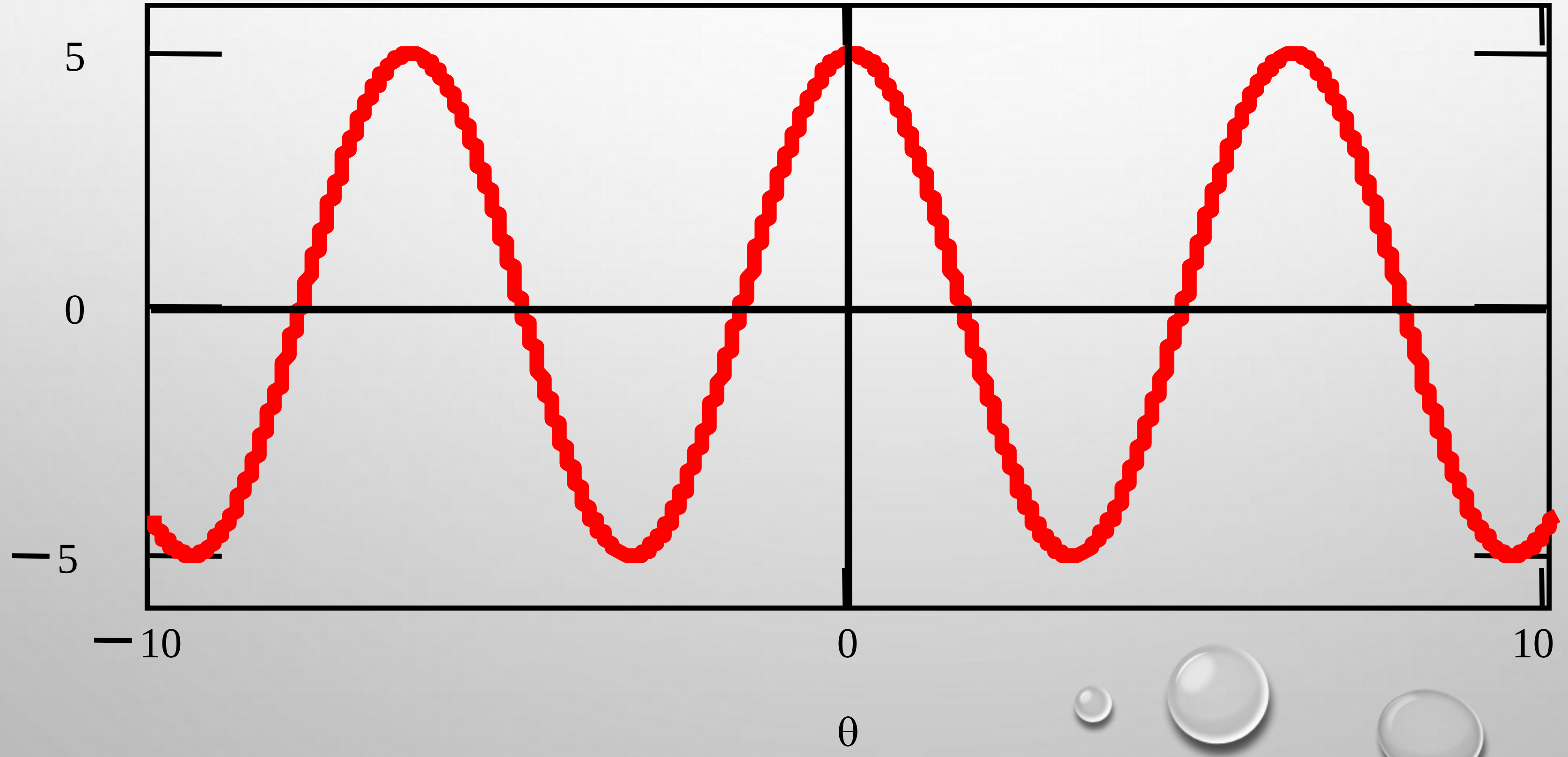
Fourier series

Even functions can solely be represented by cosine waves because, cosine waves are even functions. A sum of even functions is another even function.



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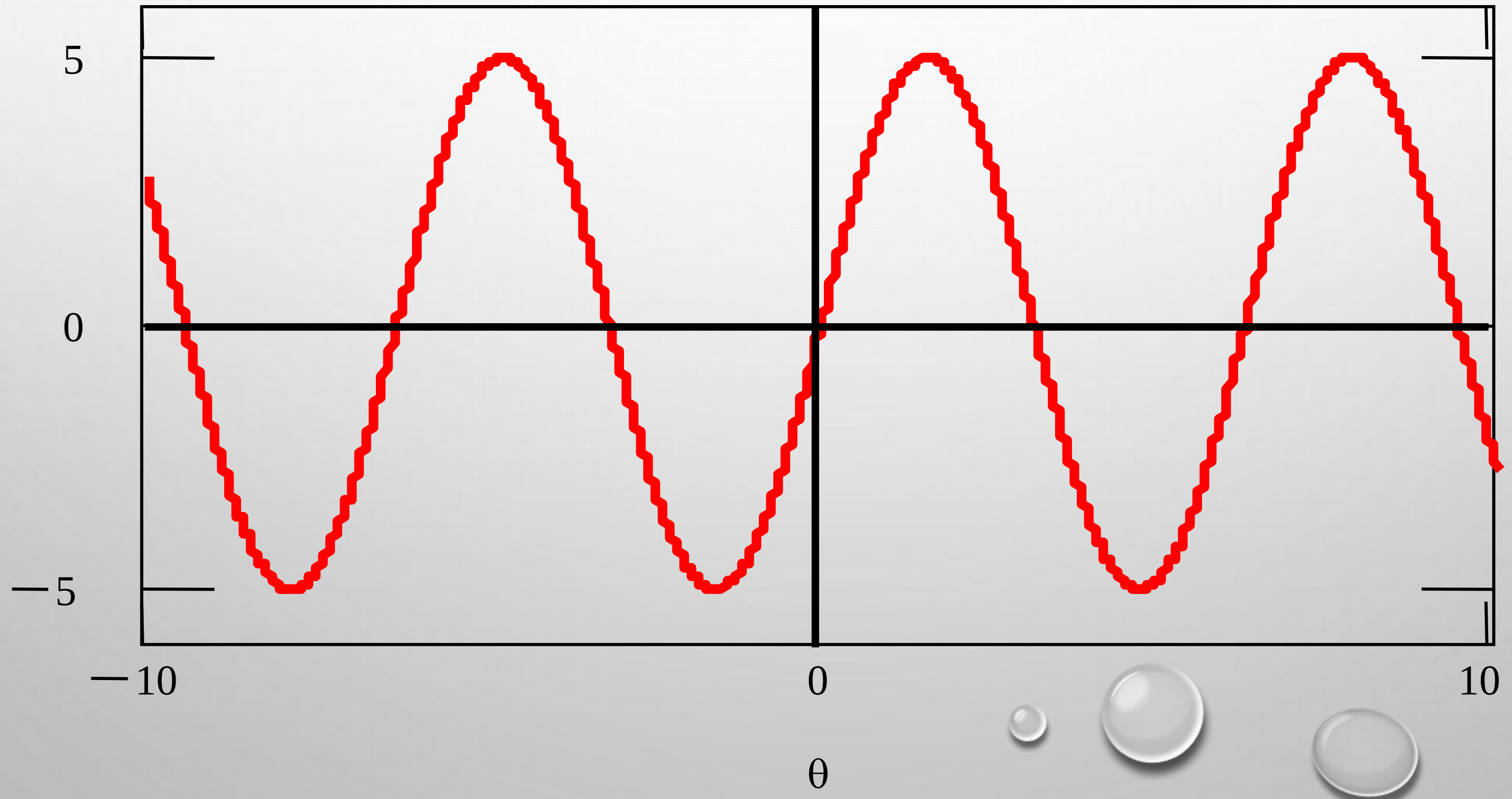
Fourier series

Odd functions can solely be represented by sine waves because, sine waves are odd functions. A sum of odd functions is another odd function.



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Fourier series

The Fourier series of an even function $\boldsymbol{f(\theta)}$ is expressed in terms of a cosine series.

$$\boldsymbol{f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\theta}$$

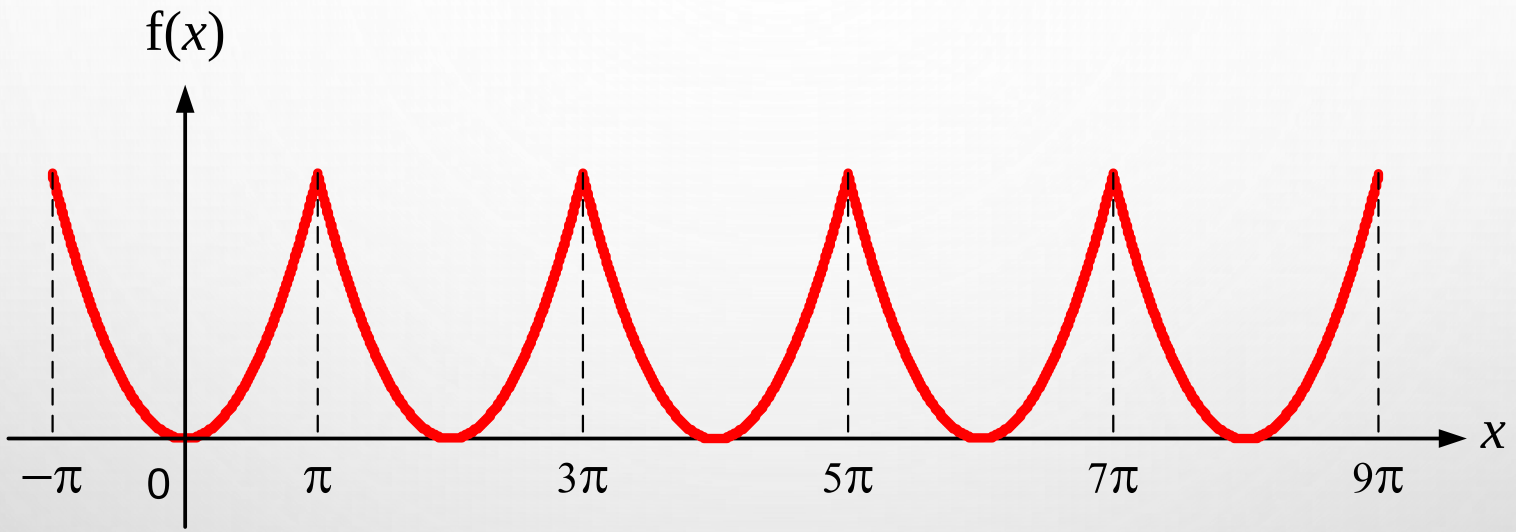
The Fourier series of an odd function $\boldsymbol{f(\theta)}$

is expressed in terms of a sine series.

$$\boldsymbol{f(\theta) = \sum_{n=1}^{\infty} b_n \sin n\,\theta}$$

Example

Find the Fourier series of the following periodic function.



$f(x) = x^2 \quad when \quad -\pi \leq x \leq \pi$

$f(\theta + 2\pi) = f(\theta)$

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Example

Sol:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\, dx$$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{x=-\pi}^{x=\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos n\, x dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x^2 \cos n\, x dx\right]$$

Use integration by parts.

Example

$$a_n = \frac{4}{n^2}\cos n\,\pi$$

$$a_n = -\frac{4}{n^2} \quad \text{when n is odd}$$

$$a_n = \frac{4}{n^2} \quad \text{when n is even}$$

This is an even function.

Therefore, $b_n = 0$

The corresponding Fourier series is

$$\frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2\,x}{2^2} + \frac{\cos 3\,x}{3^2} - \frac{\cos 4\,x}{4^2} + \cdots\right)$$

Functions Having Arbitrary Period

Assume that a function   has period T. We can relate angle (θ)with time(t) in the following manner:

$$\boldsymbol{\theta = \omega t}$$

$\boldsymbol{\omega}$ is the angular velocity in radians per second.

$$\boldsymbol{\omega = 2\pi f}$$

$\boldsymbol{f}$ is the frequency of the periodic function, $\boldsymbol{f(t)}$

Functions Having Arbitrary Period

$\boldsymbol{\theta = 2\pi f t}$ where $\boldsymbol{f = \frac{1}{T}}$

$$\boldsymbol{\theta = \frac{2\pi}{T} t}$$

Therefore,

$$\boldsymbol{\theta = \frac{2\pi}{T} t} \qquad \boldsymbol{d\theta = \frac{2\pi}{T} dt}$$

Now change the limits of integration.

$$\boldsymbol{\theta = -\pi} \qquad \boldsymbol{-\pi = \frac{2\pi}{T} t} \qquad \boldsymbol{t = -\frac{T}{2}}$$

$$\boldsymbol{\theta = \pi} \qquad \boldsymbol{\pi = \frac{2\pi}{T} t} \qquad \boldsymbol{t = \frac{T}{2}}$$

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Functions Having Arbitrary Period

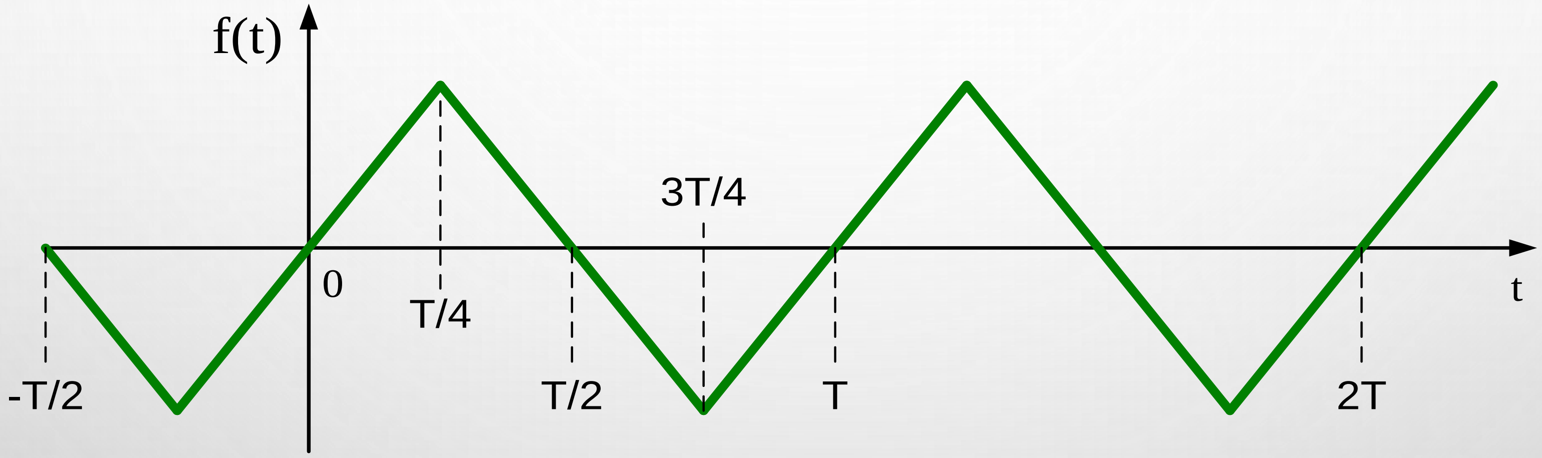
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta \quad \longrightarrow \quad a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\,\theta d\theta \quad n = 1,2,\cdots \quad \longrightarrow \quad a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\frac{2\pi n}{T}t)dt \quad n = 1,2,\cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\,\theta d\theta \quad n = 1,2,\cdots \quad \longrightarrow \quad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\frac{2\pi n}{T}t)dt \quad n = 1,2,\cdots$$

Example

Find the Fourier series of the following periodic function.



$$f(t) = t \quad when -\frac{T}{4} \le t \le \frac{T}{4} \qquad = -t + \frac{T}{2} \quad when \quad \frac{T}{4} \le t \le \frac{3T}{4}$$

$$f(t+T) = f(t)$$

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Example

This is an odd function. Therefore, $\boldsymbol{a_n = 0}$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(\frac{2\pi n}{T}t)dt$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(\frac{2\pi n}{T}t)dt$$

$$b_n = \frac{4}{T}\int_0^{T/4} t\sin(\frac{2\pi n}{T}t)dt + \quad \frac{4}{T}\int_{T/4}^{T/2}(-t+\frac{T}{2})\sin(\frac{2\pi n}{T}t)dt$$

Use integration by parts.

Example

$$b_n = \frac{4}{T}\left[2.\left(\frac{T}{2\pi n}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right] \quad = \frac{2T}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$b_n = 0$ when *n* is even.

Therefore, the Fourier series is

$$\frac{2T}{\pi^2}\left[\sin\left(\frac{2\pi}{T}t\right) - \frac{1}{3^2}\sin\left(\frac{6\pi}{T}t\right) + \frac{1}{5^2}\sin\left(\frac{10\pi}{T}t\right) - \cdots\right]$$

The Complex Form of Fourier Series

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\theta + \sum_{n=1}^{\infty} b_n \sin n\,\theta$$

Let us utilize the Euler formulae.

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2i}$$

The Complex Form of Fourier Series

The nth harmonic component of (1) can be expressed as:

$$a_n \cos n\,\theta + b_n \sin n\,\theta$$

$$= a_n \frac{e^{jn\theta} + e^{-jn\theta}}{2} + b_n \frac{e^{jn\theta} - e^{-jn\theta}}{2i}$$

$$= a_n \frac{e^{jn\theta} + e^{-jn\theta}}{2} - ib_n \frac{e^{jn\theta} - e^{-jn\theta}}{2}$$

$$= \left(\frac{a_n - jb_n}{2}\right) e^{jn\theta} + \left(\frac{a_n + jb_n}{2}\right) e^{-jn\theta}$$

The Complex Form of Fourier Series

Denoting

$$c_n = \left(\frac{a_n - jb_n}{2}\right) \quad , \quad c_{-n} = \left(\frac{a_n + jb_n}{2}\right)$$

and $c_0 = a_0$

$$a_n \cos n\,\theta + b_n \sin n\,\theta$$

$$= c_n e^{jn\theta} + c_{-n} e^{-jn\theta}$$

The Complex Form of Fourier Series

The Fourier series for $f(\theta)$ can be expressed as:

$$f(\theta) = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{jn\theta} + c_{-n}e^{-jn\theta}\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\theta}$$

The coefficients can be evaluated in the following manner.

$$c_n = \left(\frac{a_n - jb_n}{2}\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\,\theta d\theta - \frac{j}{2\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\,\theta d\theta$$

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The Complex Form of Fourier Series

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)(\cos n\,\theta - j\sin n\,\theta)d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-jn\theta}d\theta$$

$$c_{-n} = \left(\frac{a_n + jb_n}{2}\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\,\theta d\theta + \frac{j}{2\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\,\theta d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)(\cos n\,\theta + j\sin n\,\theta)d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{jn\theta}d\theta$$

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The Complex Form of Fourier Series

$c_0 = \left(\frac{a_0}{2}\right)$ $c_n = \left(\frac{a_n - jb_n}{2}\right)$ $c_{-n} = \left(\frac{a_n + jb_n}{2}\right)$

Note that c_{-n} is the complex conjugate of c_n . Hence we may write that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-jn\theta}d\theta \qquad n = 0,\ \pm1,\ \pm2,\ \cdots$$

or

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x).\,e^{-inx}dx$$

The complex form of the Fourier series of $f(\theta)$ with period 2π is:

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{jn\theta}$$

Example

Using complex form, find the Fourier series of the function

$$f(\theta) = \begin{cases} -1 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}$$

Sol:

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,dx + \int_{0}^{\pi} dx\right] = \frac{1}{2\pi}\left[(-x)\big|_{-\pi}^{0} + x\big|_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}(-\not{\pi} + \not{\pi}) = 0,$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,e^{-inx}dx + \int_{0}^{\pi} e^{-inx}dx\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\left(e^{-inx}\right)\big|_{-\pi}^{0}}{-in} + \frac{\left(e^{-inx}\right)\big|_{0}^{\pi}}{-in}\right] = \frac{i}{2\pi n}\left[-\left(1 - e^{in\pi}\right) + e^{-in\pi} - 1\right]$$

$$= \frac{i}{2\pi n}\left[e^{in\pi} + e^{-in\pi} - 2\right] = \frac{i}{\pi n}\left[\frac{e^{in\pi} + e^{-in\pi}}{2} - 1\right] = \frac{i}{\pi n}\left[\cos n\pi - 1\right]$$

$$= \frac{i}{\pi n}\left[(-1)^n - 1\right].$$

Ruaa Shallal Abbas