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DIGITAL SIGNAL PROCESSING

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LINEAR DIFFERENCE EQUATIONS

1. First order homogeneous equation:

We consider an equation of the form:

$$x_{n+1} - ax_n = 0$$

where x_n , n = 0; 1; 2; 3; are unknown and a is fixed constant. This equation is called a ***first order homogeneous equation***.

The term homogeneous means that the right hand side is zero.

The general solution is $x_n = \alpha^n x_0$

Where x_0 is called the ***initial value***

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EXAMPLE

Find the function $\mathbf{f(\ x,y) = 2\,x\ +\ y}$ is homogeneous or not?

Sol:

$$\mathbf{f(\alpha x,\ \alpha y) = 2(\alpha x)+(\alpha y)}$$

$$\mathbf{=\alpha(2\,x\ +\ y)}$$

$$\mathbf{=\alpha\ f(x,\ y)}$$

The function is homogeneous of degree 1

EXAMPLE

Find the function **f(x,y) =** $\boldsymbol{x^2 + y^2}$ is homogeneous or not?

Sol:

$$\mathbf{f}(\alpha \mathbf{x}, \alpha \mathbf{y}) = (\alpha x)^2 + (\alpha y)^2$$

$$= \alpha^2 (x^2 + y^2)$$

$$= \alpha^2 \, \mathbf{f}(\mathbf{x}, \mathbf{y})$$

The function is homogeneous of degree 2

EXAMPLE

Find the function **f(x,y) =** x^3 **-** y^2 is homogeneous or not?

Sol:

$$f(\alpha x, \alpha y) = (\alpha x)^3 - (\alpha y)^2$$

$$= \alpha^3 x^3 + \alpha^2 y^2$$

The function is not homogeneous because which does not equal α^n **f(x,y)** for any n.

EXAMPLE

Solve the following differential equation, the initial value of x(0)=3

x(n+1) = - x(n)

Sol:

we can write the equation as :

x(n+1)+ x(n)=0

This the first order homogeneous equation

So the solution will be as:

$$x_n = \alpha^n x_0$$

$$= (-1)^n *3$$

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LINEAR DIFFERENCE EQUATIONS

2. First order inhomogeneous equation:

Let us consider an equation of the form:

$$\boldsymbol{a x_n} + \mathbf{b}\, \boldsymbol{x_{n-1}} = \boldsymbol{c}$$

where $\boldsymbol{c_n}$ is a **given sequence** and $\boldsymbol{x_n}$ is unknown. This equation is called ***inhomogeneous equation*** because the term $\boldsymbol{c_n}$.

The general solution of $\boldsymbol{a x_n} + \mathbf{b}\, \boldsymbol{x_{n-1}} = \boldsymbol{c}$ is:

$$\boldsymbol{x_n} = \boldsymbol{a^n}\, \boldsymbol{x_0} + \mathbf{c}\, \frac{\boldsymbol{a^n - 1}}{\boldsymbol{a - 1}}$$

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EXAMPLE

Solve the following differential equation, the initial value of x(0)=3

-x(n+1) = - x(n)+4

Sol:

we can write the equation as :

-x(n+1)+ x(n)=4

This the first order inhomogeneous equation

So the solution will be as:

$$x_n = a^n\, x_0 + c\,\frac{a^n - 1}{a - 1}$$

$$= (-1)^n *3 + 4* \frac{(-1)^n - 1}{-1-1} = (-1)^n *3 - 2\,[(-1)^n - 1]$$

$$= (-1)^n + 2$$

EXAMPLE

Solve the following differential equation, the initial value of x(0)=5

$$\mathbf{x(n+1) = 2\,x(n)+n}$$

Sol:

we can write the equation as :

$$\mathbf{x(n+1)- 2x(n)=n}$$

This the first order inhomogeneous equation

So the solution will be as:

$$\boldsymbol{x_n = a^n\, x_0 + c\,\frac{a^n-1}{a-1}}$$

$$= \mathbf{(2)^n *5 + n* \frac{(2)^n-1}{2-1} = (2)^n *5 + n((2)^n -1)}$$

LINEAR CONSTANT COEFFICIENT DIFFERENCE EQUATIONS

A subclass of linear shift invariant systems are those for which the input x(n) and output y(n) satisfy an N^{th} order linear constant coefficient difference equation, given by

$$\sum_{k=0}^{N} a_k \, y(n-k) = \sum_{r=0}^{M} b_r \, x(n-r) \;, a_0 \neq 0$$

If the system is causal, then we can rearrange the equation above to provide a computational realization of the system as follows:

$$y(n) = -\sum_{k=1}^{N} \frac{a_k}{a_0} \, y(n-k) + \sum_{r=0}^{M} \frac{b_r}{a_0} \, x(n-r)$$

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EXAMPLE

Solve the following difference equation for y(n), assuming y(n) = 0 for all $n < 0$ and $x(n) = \delta(n)$.

$$y(n) - a\,y(n-1) = x(n)$$

This corresponds to calculate the response of the system when excited by an impulse, assuming zero initial conditions.

<u>Solution:</u>

$y(n) - a\,y(n-1) = x(n)$

Evaluating when $n = 0$, $y(0) = a\,y(-1) + x(0) = 1$

$n = 1$, $y(1) = a\,y(0) + x(1) = a$

$n = 2$, $y(2) = a\,y(1) + x(2) = a^2$

Continuing this process it is easy to see for all $n \geq 0$ that $\mathbf{y(n) = a^n}$

Since the response of the system for $n < 0$ is defined to be zero, the unit sample **<u>RESPONSE</u>** becomes

$$\mathbf{h(n) = a^n\,u(n)}$$

EXAMPLE

Given the first order linear constant coefficient difference equation

$$y(n) = a\ y(n-1) + x(n)$$

find the impulse response of the system by first finding the system function H(z) and then taking the inverse transform. Assuming the system is causal.

Solution:

$y(n) = a\ y(n-1) + x(n)$

$Y(z) = a\ z^{-1}Y(z) + X(z) \quad \rightarrow \quad Y(z)(1 - a\ z^{-1}) = X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-a\,z^{-1}} = \frac{z}{z-a}$$

By the causality requirement, the inverse transform of H(z) is a positive time exponential and with the ROC z > a, then, h(n) becomes

$$\mathbf{h(n) = z^{-1}\{\frac{z}{z-a}\} = a^n\ u(n)}$$

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TRANSFER FUNCTION (SYSTEM FUNCTION)

The output y(n) of LTI system to an input sequence x(n) can be obtained by computing the convolution of x(n) with the unit sample (Impulse) response h(n) of the system.

$$y(n) = h(n) * x(n)$$

Expressing this relationship in the z-domain as

$$Y(z) = H(z)\,X(z)$$

Impulse Response:

$$\mathbf{H(z)} = \frac{\mathbf{Y(z)}}{\mathbf{X(z)}}$$

The LTI system can be described by means of a constant coefficient linear difference equation as follows:

TRANSFER FUNCTION (SYSTEM FUNCTION)

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

$$Y(z) = \sum_{k=0}^{N} b(k)z^{-k}X(z) - \sum_{k=1}^{M} a(k)z^{-k}Y(z)$$

The transfer function of the LTI system,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{1 + \sum_{k=1}^{M} a(k)z^{-k}}$$

EXAMPLE

Find the impulse response of the system **y(n) = x(n+1) + x(n) + x(n-1)**, then find the system function in z-domain.

Solution:

y(n) = x(n+1) + x(n) + x(n-1)

Y(z) = X(z) z^1 + X(z) + X(z) z^{-1}

Y(z) = X(z) [z^1 +1 + z^{-1}]   **system response**

H(z) = $\frac{Y(z)}{X(z)}$ = z^1 +1 + z^{-1} **system function**

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EXAMPLE

Find the difference equation realization and the response of the system represented by the following causal system function $H(z) = \frac{(z+1)}{(z^2-2z+3)}$

Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(z+1)}{(z^2-2z+3)} = \frac{z^{-1}+z^{-2}}{1-2z^{-1}+3z^{-2}} \;;$$

By taking the z^{-1} yields the difference equation

y(n) – 2y(n-1) + 3y(n-2) = x(n-1) + x(n-2)

y(n) = 2y(n-1) - 3y(n-2) + x(n-1) + x(n-2)

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