FURAT AL AWSAT TECHNICAL UNIVERSITY
NAJAF COLLEGE OF TECHNOLOGY
DEPARTMENT OF AVIONICS ENGINEERING

DIGITAL SIGNAL PROCESSING

3rd YEAR

BY

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Z-TRANSFORM

- **The z-transform** converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency-domain. It is a very important tool in describing and analyzing digital systems. It also offers the techniques for digital filter design and frequency analysis of digital signals.

- The z-transform of a sequence x(n), designated by X(z) or Z(x(n)), is defined as:

$$X(z) = Z\big(x(n)\big) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

- Where, z is the complex variable, n is an integer time index.

$z = r e^{j\omega}$

r: magnitude

w: Angle or Phase

$$z = r(cos\omega + jsin\omega)$$

- $X(z) = \cdots + x(-1)z^{1} + x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \cdots$

Z-TRANSFORM

- When x(n) is defined for n≥0,i.e. causal, one sided Z transform is given by:

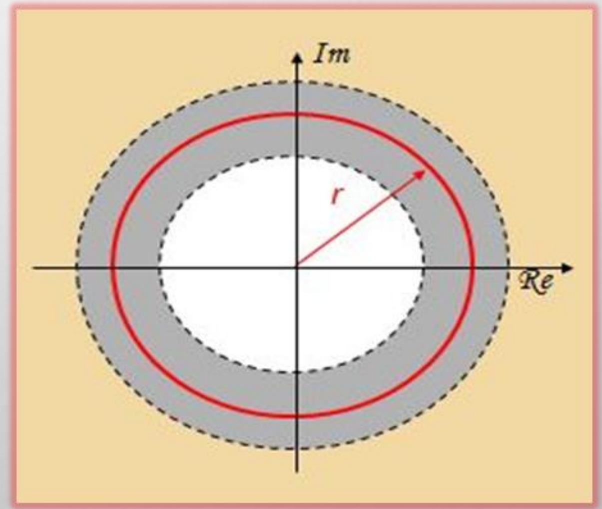
$$X(z) = Z\big(x(n)\big) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Region of Convergence (ROC)

Roc : represent the region that in which the function be finite.

or

it values of (z) that make the values of X[z] Finite values



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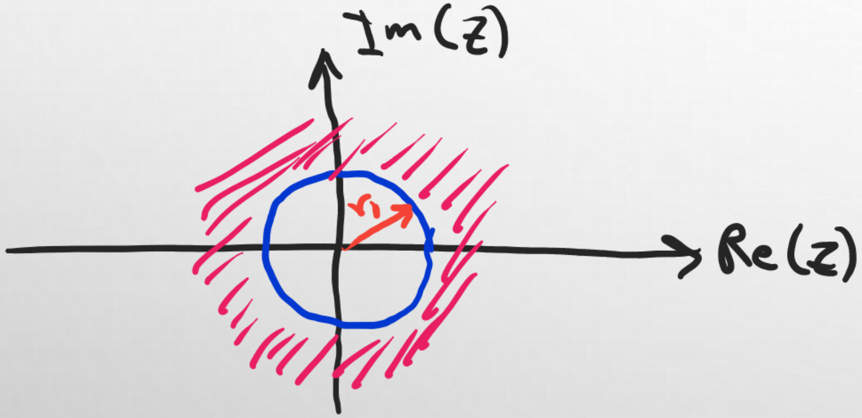
Notes on Roc :-

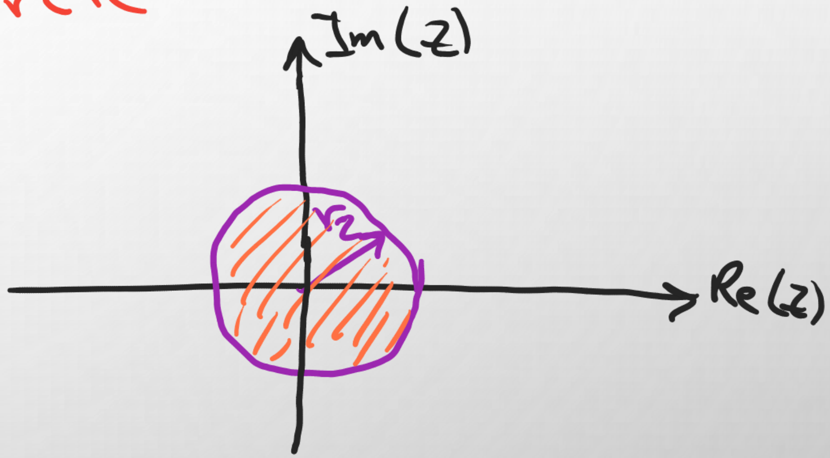
① if the signal be equal zero when $n<0$, Roc be always out of circle



Roc: $|z| > r_1$

ROC: $|z| > |a| \Longleftrightarrow$ causal system

② If the signal be equal zero when $n>0$, Roc be inside the circle



Roc: $|z| < r_2$

ROC: $|z| < |a| \Longleftrightarrow$ anti-causal system

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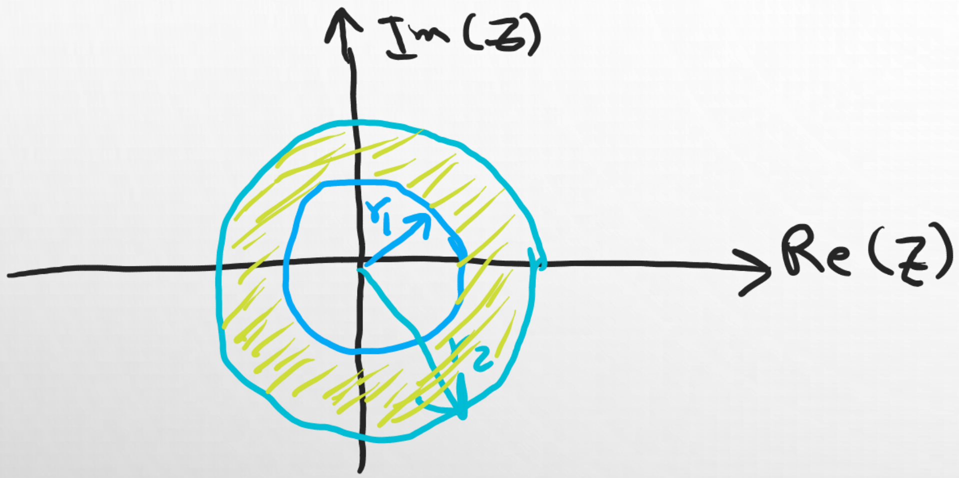
③ If the signal defined in negative & positive parts, Roc be as showen below



ROC: $|b| < |z| < |a| \iff$ two-sided system (non-causal)

Roc : $r_1 < |z| < r_2$

Ex/ Find $X[z]$ & RoC of the function $x[n] = u[n]$

Sol.

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} (z^{-1})^{n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

Note

If we have a sequence written as: $1 + a + a^2 + a^3 + \cdots$

we can write this sequence as: $\dfrac{\text{First term}}{1 - \text{Comm. ratio}} = \dfrac{1}{1-a}$

with condition $|C.R.| < 1$

$|a| < 1$

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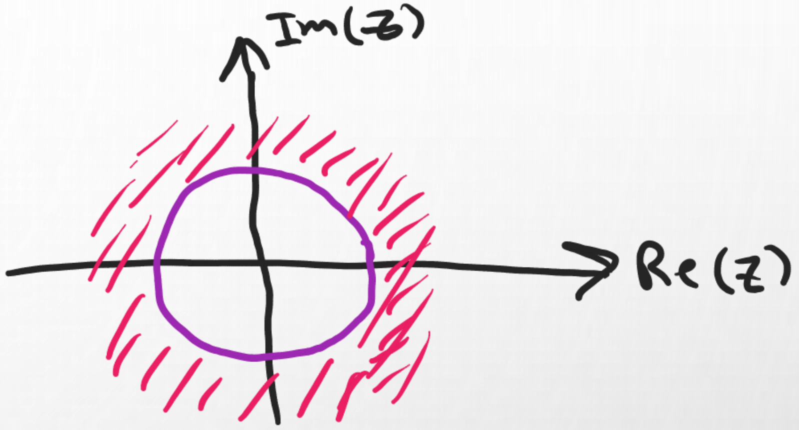
$$\therefore\ X[z] = 1 + (z^{-1})^1 + (z^{-1})^2 + (z^{-1})^3 + \cdots$$

$$\therefore\ X[z] = \frac{1}{1-z^{-1}} = \left.\frac{1}{1-\frac{1}{z}}\right\} * z$$

$$\therefore\ X[z] = \frac{z}{z-1}$$

$$|z^{-1}| < 1 \Rightarrow |z| > 1$$



RoC : $|z| > 1$

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Ex

let $x[n] = a^n u[n]$ Calculate $X[z]$ and Roc ?

Sol.

$$X[z] = \sum_{n=-\infty}^{\infty} a^n u[n] \cdot z^{-n}$$

$$\therefore X[z] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= 1 + a z^{-1} + (a z^{-1})^2 + \cdots$$

$$\therefore C.R = a \cdot z^{-1}$$

$$\therefore X[z] = \frac{1}{1 - a z^{-1}}$$

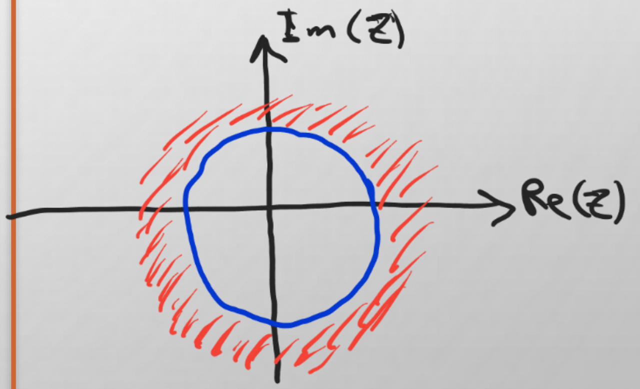
$$= \frac{1}{1 - \frac{a}{z}} \Big] * z$$

$$= \frac{z}{z - a}$$

Roc : $|a z^{-1}| < 1$

$$\left|\frac{a}{z}\right| < 1$$

$$\therefore |z| > a$$



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	Signal, $x[n]$	Z-transform, $X(z)$	ROC
1	$\delta[n]$	1	all z
2	$\delta[n-n_0]$	z^{-n_0}	$z \neq 0$
3	$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
4	$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
5	$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\|z\| > 1$
6	$-nu[-n-1]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\|z\| < 1$
7	$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\|z\| > 1$
8	$-n^2u[-n-1]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\|z\| < 1$
9	$n^3u[n]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\|z\| > 1$
10	$-n^3u[-n-1]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\|z\| < 1$
11	$a^nu[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
12	$-a^nu[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
13	$na^nu[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
14	$-na^nu[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
15	$n^2a^nu[n]$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\|z\| > \|a\|$
16	$-n^2a^nu[-n-1]$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\|z\| < \|a\|$
17	$\binom{n+m-1}{m-1}a^nu[n]$	$\frac{1}{(1-az^{-1})^m}$, for positive integer m[13]	$\|z\| > \|a\|$
18	$(-1)^m\binom{-n-1}{m-1}a^nu[-n-m]$	$\frac{1}{(1-az^{-1})^m}$, for positive integer m[13]	$\|z\| < \|a\|$
19	$\cos(\omega_0 n)u[n]$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\|z\| > 1$
20	$\sin(\omega_0 n)u[n]$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\|z\| > 1$
21	$a^n\cos(\omega_0 n)u[n]$	$\frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\|z\| > \|a\|$
22	$a^n\sin(\omega_0 n)u[n]$	$\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\|z\| > \|a\|$

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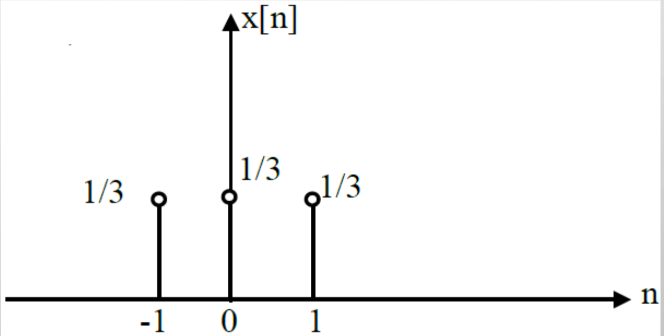
Example: Find Z-transform for the unit impulse sequence

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = z^0 = 1. \quad \text{ROC: All } z$$

Example: Find the **Z-transform** of the signal shown below:

Solution:



$$X(z) = \sum_{n=-1}^{1} x[n] z^{-n}$$
$$= \frac{1}{3} z + \frac{1}{3} + \frac{1}{3} z^{-1}$$

Example: Find the Z-transform of the second order recursive filter, given:

$$h[n] = \begin{cases} r^n \cos(w_0 n) & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$H(z) = \sum_{n=0}^{\infty} r^n \cos(w_0 n) z^{-n} \qquad = \sum_{n=0}^{\infty} r^n \frac{e^{jw_0 n} + e^{-jw_0 n}}{2} z^{-n}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \left(re^{jw_0} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(re^{-jw_0} z^{-1}\right)^n\right] \qquad = \frac{1}{2}\left[\frac{1}{1 - re^{jw_0} z^{-1}} + \frac{1}{1 - re^{-jw_0} z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1 - re^{-jw_0} z^{-1} + 1 + re^{jw_0} z^{-1}}{(1 - re^{jw_0} z^{-1})(1 - re^{-jw_0} z^{-1})}\right] \qquad = \frac{1 - r\cos(w_0) z^{-1}}{1 - 2r\cos(w_0) z^{-1} + r^2 z^{-2}}$$

Z-Transform Properties

1) **Linearity**

$$Ax_1[n] + B\,x_2[n] \leftrightarrow AX_1[z] + BX_2[z]$$

2) **Time-shifting**

$$x[n-a] \leftrightarrow X(z)z^{-a}$$

3) **Convolution**

$$x_1[n] \otimes x_2[n] \leftrightarrow X_1[z].X_2[z]$$

4) **Differentiation**

$$nx[n] \leftrightarrow (-z)\frac{dX(z)}{dz}$$

5) **Multiplication by an exponential sequence:**

$$a^n x[n] \leftrightarrow X\left(\frac{z}{a}\right)$$

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Example: Determine the **Z-transform** of the sequence given by: **y[n] = (n+1) α^n u[n]**

Solution:

$$y(n) = \alpha^n u[n] + n\alpha^n u[n]$$

$$\because \alpha^n u[n] \Leftrightarrow \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$$

$$Y(z) = \frac{1}{1-\alpha z^{-1}} + (-z)\frac{d}{dz}\frac{1}{1-\alpha z^{-1}}$$

$$= \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} = \frac{1}{(1-\alpha z^{-1})^2}$$