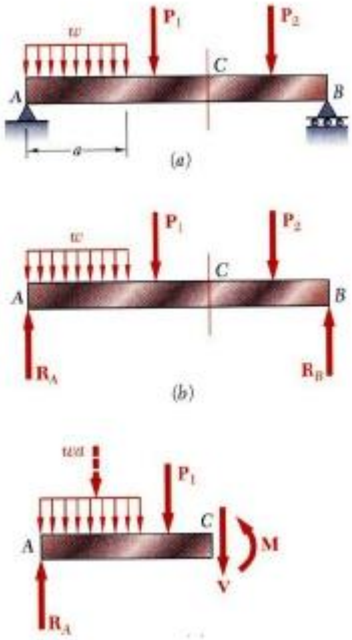
CHAPTER 5

BEAMS

Introduction :



- *Beams* - structural members supporting loads at various points along the member.
- Transverse loadings of beams are classified as *concentrated* loads or *distributed* loads.
- Applied loads result in internal forces consisting of a shear force (from the shear stress distribution) and a bending couple (from the normal stress distribution).

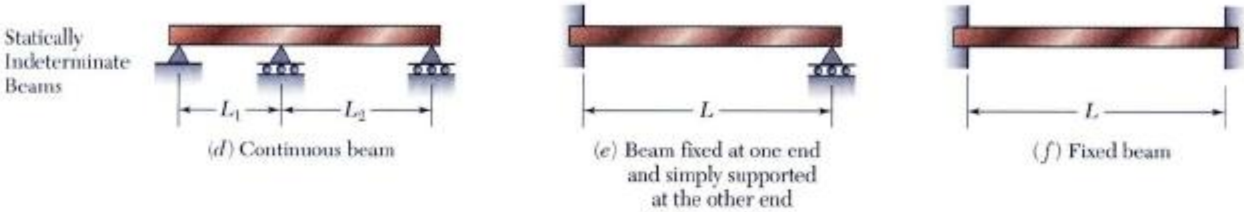
Classification of Beams:

1- Statically Determinate Beams:

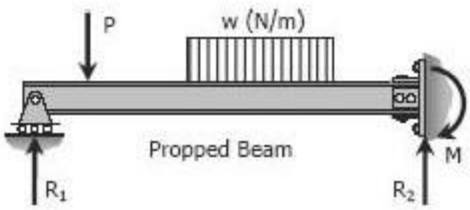
Statically determinate beams are those beams in which the reactions of the supports may be determined by the use of the equations of static equilibrium. The beams shown below are examples of statically determinate beams.

2- Statically Indeterminate Beams:

If the number of reactions exerted upon a beam exceeds the number of equations in static equilibrium, the beam is said to be statically indeterminate. In order to solve the reactions of the beam, the static equations must be supplemented by equations based upon the elastic deformations of the beam.

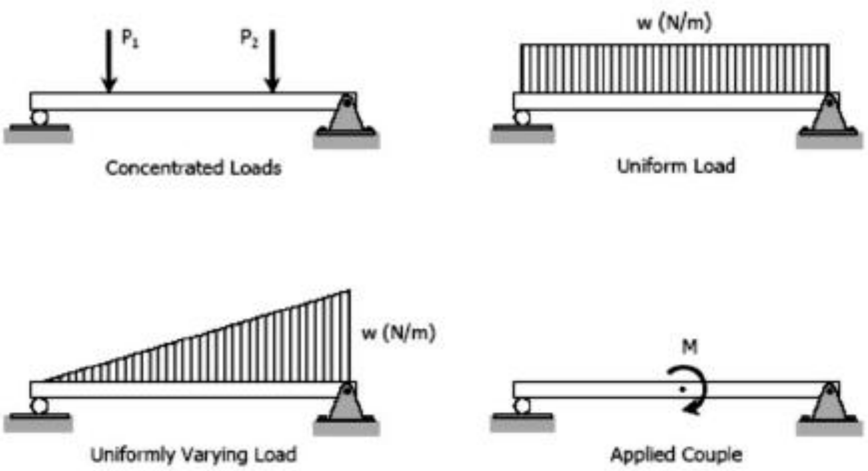


The degree of indeterminacy is taken as the difference between the number of reactions to the number of equations in static equilibrium that can be applied. In the case of the propped beam shown, there are three reactions (R_1, R_2, and M) while only two equations ($\Sigma M = 0$ and $\Sigma F_v = 0$) can be applied, thus the beam is indeterminate to the first degree ($3 - 2 = 1$).



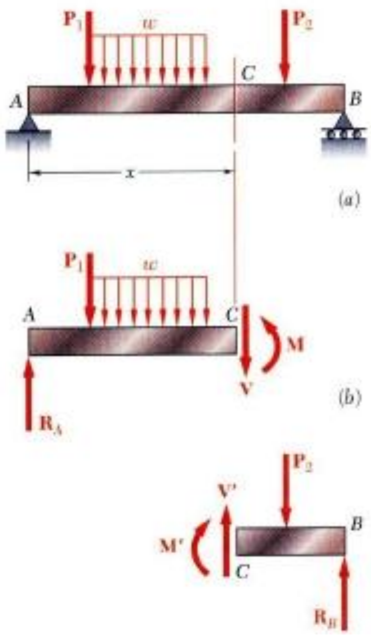
TYPES OF LOADING

Loads applied to the beam may consist of a concentrated load (load applied at a point), uniform load, uniformly varying load, or an applied couple or moment. These loads are shown in the following figures.



Shear Force and Bending Moment Diagrams

Shear Force and Bending Moment Diagrams are plots of the shear forces and bending moments, respectively, along the length of a beam. The purpose of these plots is to clearly show maximum of the shear force and bending moment, which are important in the design of beams.



The most common sign convention for the shear force and bending moment in beams is shown in Fig. 9.12.

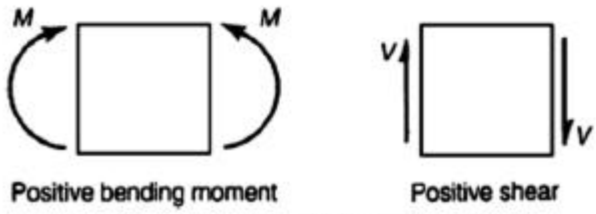


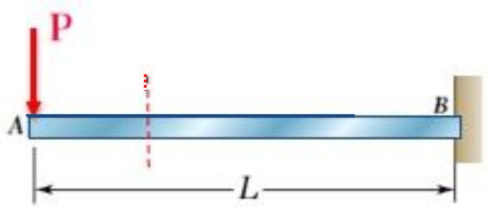
Figure 9.12 Sign convention for bending moment and shear

One method of determining the shear and moment diagrams is by the following steps:

1. Determine the reactions from equilibrium of the entire beam.
2. Cut the beam at an arbitrary point.
3. Show the unknown shear and moment on the cut using the positive sign convention shown in Fig. 9.12.
4. Sum forces in the vertical direction to determine the unknown shear.
5. Sum moments about the cut to determine the unknown moment.

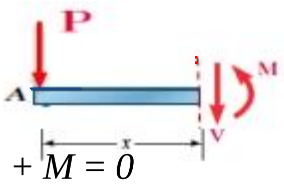
Example (1)

For the beam shown, derive equations for shear force and bending moment at any point along the beam.



Solution:

We cut the beam at a point between A and B at distance x from A and draw thefree-body diagram of the left part of the beam, directing **V** and **M** as indicated in the figure.



$\Sigma F_y = 0$:

$$P + V = 0$$

$$V = -P \quad (\downarrow)$$

$\Sigma M_x = 0$:

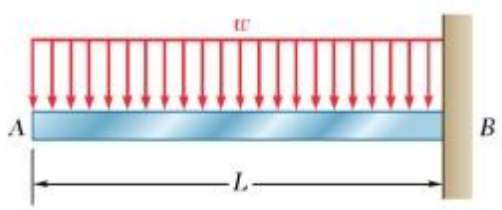
$$P.x + M = 0$$

$$M = -Px(\circlearrowright)$$

Note** that shear force is constant (equal P) along the beam, and bending moment is a linear function of (* ***x *).*

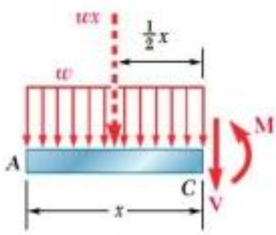
Example (2):

For a cantilever beam AB of span L supporting a uniformly distributed load w, derive equations for shear force and bending moment at any point along the beam.



Solution:

We cut the beam at a point C between A and B and draw free-body diagram of AC, directing **V** and **M** as indicated in Fig. Denoting by x the distance from A to C and replacing the distributed load over AC by its resultant (wx) applied at the mid point of AC, we write:



$\Sigma F_y = 0$:

$$-wx - V = 0$$

$$V = -wx$$

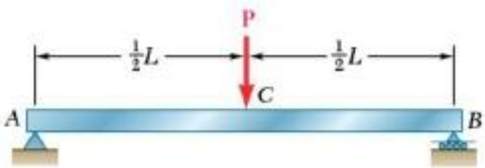
$\Sigma M_x = 0$:

$$wx\left(\frac{x}{2}\right) + M = 0$$

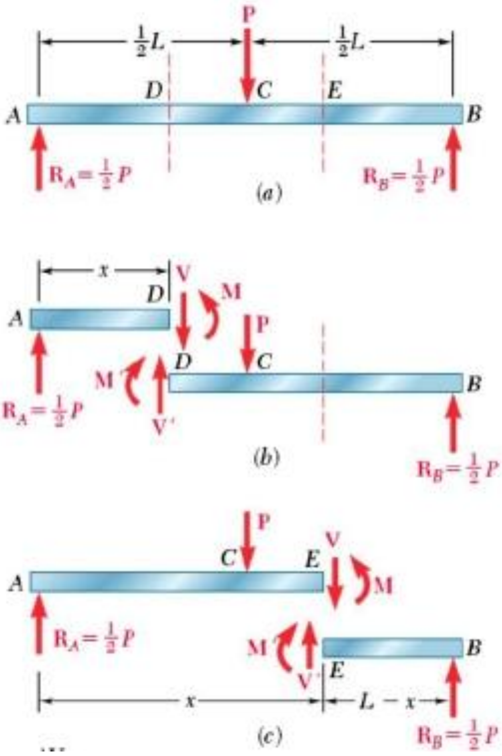
$$\boxed{M = -\frac{1}{2}wx^2}$$

Example (3):

For the simply supported beam ***AB*** of span ***L*** supporting a single concentrated load ***P***, derive equations for shear force and bending moment at any point along the beam.



Solution:



We first determine the reactions at the supports from the free-body diagram of the entire beam; we find that the magnitude of each reaction is equal to $P/2$. Next we cut the beam at a point D between A and C and draw the free-body diagrams of AD and DB. *Assuming that shear and bending moment are positive*, we direct the internal forces **V** and **V'** and the internal couples **M** and **M'** as indicated in Fig. Considering the free body AD and writing that the sum of the vertical components and the sum of the moments about D of the forces acting on the free body are zero, we find:

$V = +P/2$ and $M = +Px/2$.

Both the shear and bending moment are therefore positive; this may be checked by observing that the reaction at A tends to shear off and to bend the beam at D as indicated in Figs. *b* and *c*. The shear has a constant value $V = P/2$, while the bending moment increases linearly from $M = 0$ at $x = 0$ to $M = PL/4$ at $x = L/2$.

Cutting, now, the beam at a point E between C and B and considering the free body EB (Fig. *c*), we write that the sum of the vertical components and the sum of the moments about E of the forces acting on the free body are zero. We obtain:

$V = -P/2$ and $M = P(L - x)/2$.

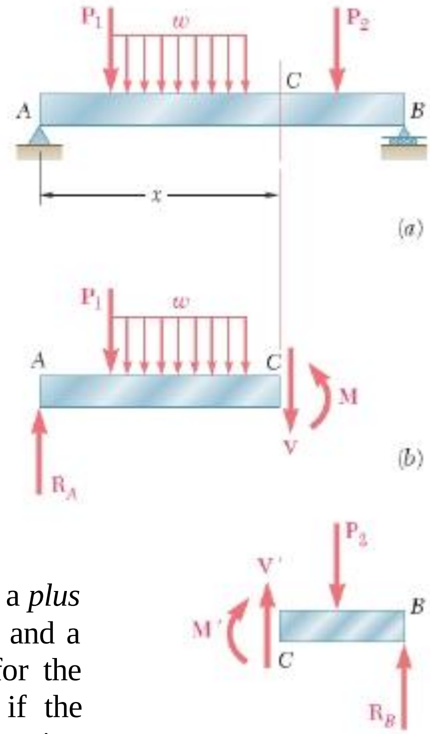
The shear is therefore negative and the bending moment positive; this can be checked by observing that the reaction at B bends the beam at E as indicated in Fig. *c* but tends to shear it off in a manner opposite to that shown in Fig. *b*.

***Note** that the shear has a constant value $V = -P/2$ between C and B, while the bending moment decreases linearly from $M = PL/4$ at $x = L/2$ to $M = 0$ at $x = L$.*

Shear Force and Moment Diagram

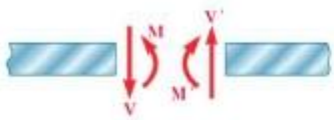
The determination of the maximum absolute values of the shear and of the bending moment in a beam are greatly facilitated if *V* and *M* are plotted against the distance *x* measured from one end of the beam. Besides, as you will see later, the knowledge of *M* as a function of *x* is essential to the determination of the deflection of a beam.

In the examples and sample problems of this section, the shear and bending-moment diagrams will be obtained by determining the values of *V* and *M* at selected points of the beam. These values will be found in the usual way, i.e., by passing a section through the point where they are to be determined (Fig. *a*) and considering the equilibrium of the portion of beam located on either side of the section (Fig. *b*). Since the shear forces **V** and **V** have opposite senses, recording the shear at point *C* with an up or down arrow would be meaning less, unless we indicated at the same time which of the free bodies *AC* and *CB* we are considering.



For this reason, the shear *V* will be recorded with a sign: a *plus sign* if the shearing forces are directed as shown in Fig.*a*, and a *minus sign* otherwise. A similar convention will apply for the bending moment *M.* It will be considered as positive if the bending couples are directed as shown in that figure, and negative otherwise. Summarizing the sign conventions we have presented, we state:

The shear V and the bending moment M at a given point of abeam are said to be positive when the internal forces and couples actingon each portion of the beam are directed as shown in Fig. a.



(*a*) Internal forces
(positive shear and positive bending moment)

Example (4): For the beam shown, plot the shear and moment diagram.

Solution:

First, solve for the unknown reactions using the free-body diagram of the beam shown in Fig, (a). to find the reactions, sum moments about the left end which gives:

$$6R_2 - (3)(2) = 0 \text{ or } R_2 = 6/6 = 1 \text{ kN}$$

Sum forces in the vertical direction to get:

$$R_1 + R_2 = 3 = R_1 + 1 \text{ or } R_1 = 2 \text{ kN}$$

Cut the beam between the left end and the load as shown in (b). Show the unknown moment and shear on the cut using the positive sign convention. Sum the vertical forces to get:

$V = 2$ kN (independent of x)

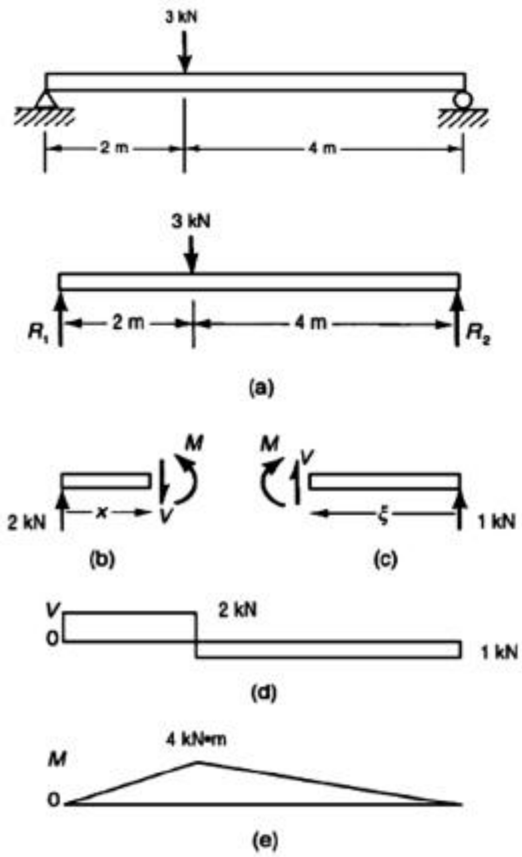
Sum moments about the cut to get:

$M = R_1x = 2x$

Repeat the procedure by making a cut between the right end of the beam and the 3-kN load, as shown in (c). Again, sum vertical forces and sum moments about the cut to get:

$V = 1$ kN (independent of x), and $M = 1x$

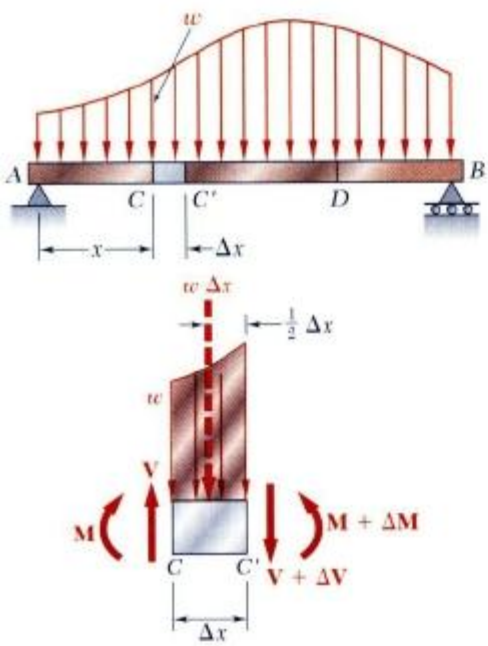
The plots of these expressions for shear and moment give the shear and moment diagrams (as shown in Fig.(d) and (e).



It should be noted that the shear diagram in this example has a jump at the point of the load and that the jump is equal to the load. This is always the case. Similarly, a moment diagram will have a jump equal to an applied concentrated moment. In this example, there was no concentrated moment applied, so the moment was everywhere continuous.

Another useful way of determining the shear and moment diagram is by using differential relationships. These relationships are found by considering an element of length Δ*x* of the beam.

The forces on that element are shown in Fig.



$$\sum F_y = 0 : \quad V - (V + \Delta V) - w\Delta x = 0$$

$$\Delta V = -w\Delta x$$

Summation of forces in the *y* direction gives :

$$\frac{dV}{dx} = -w \qquad , \qquad V_D - V_C = -\int_{x_C}^{x_D} w\,dx$$

$$\sum M_C = 0 : \quad , \quad (M + \Delta M) - M - V\Delta x + w\Delta x\frac{\Delta x}{2} = 0$$

$$\Delta M = V\Delta x - \frac{1}{2}w(\Delta x)^2 \quad , \quad \frac{dM}{dx} = V$$

$$M_D - M_C = \int_{x_C}^{x_D} V\,dx$$

The load $q = -w_0$, so Eq. (9.59) reads

$$V = V_0 - \int_0^x w_0\,dx = \frac{w_0 L}{2} - w_0 x$$

Noting that the moment at $x = 0$ is zero, Eq. (9.60) gives

$$M = M_0 - \int_0^x \left(\frac{w_0 L}{2} - w_0 x\right) dx = 0 + \frac{w_0 L x}{2} - \frac{w_0 x^2}{2} = \frac{w_0 x}{2}(L - x)$$

Example(5) The simply supported uniform beam shown in Exhibit 16 carries a uniform load of **w0** . Plot the shear and moment diagrams for this beam.

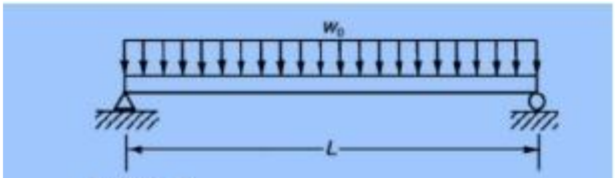


Exhibit 16

Solution

As before, the reactions can be found first from the free-body diagram of the beam shown in Exhibit 17(a). It can be seen that, from symmetry, $R_1 = R_2$. Summing vertical forces then gives

$$R = R_1 = R_2 = \frac{w_0 L}{2}$$

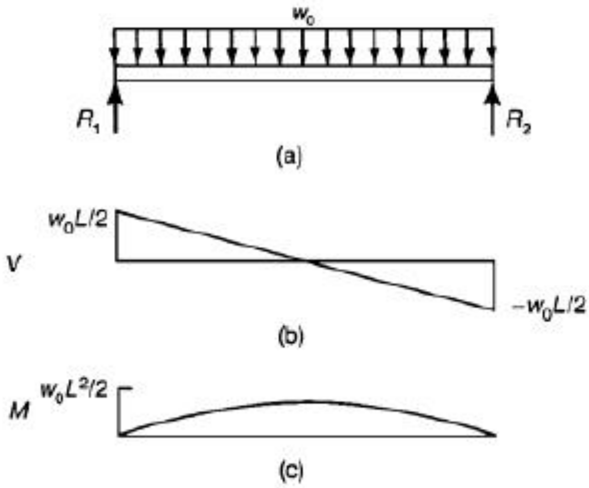


Exhibit 17

It can be seen that the shear diagram is a straight line, and the moment varies parabolically with *x*. Shear and moment diagrams are shown in Exhibit 17(b) and Exhibit 17(c). It can be seen that the maximum bending moment occurs at the center of the beam where the shear stress is zero. The maximum bending moment always has a relative maximum at the place where the shear is zero because the shear is the derivative of the moment, and relative maxima occur when the derivative is zero.

Solved problems

Write shear and moment equations for the beams in the following problems. In each problem, let x be the distance measured from left end of the beam. Also, draw shear and moment diagrams, specifying values at all change of loading positions and at points of zero shear. Neglect the mass of the beam in each problem.

Problem 403

Beam loaded as shown in Fig. P-403.

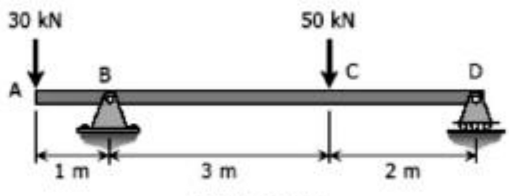


Figure P-403

Solution:

From the load diagram:

$\Sigma M_B = 0$

$5R_D + 1(30) = 3(50)$

$R_D = 24$ kN

$\Sigma M_D = 0$

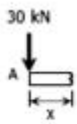
$5R_B = 2(50) + 6(30)$

$R_B = 56$ kN

Segment AB:

$V_{AB} = -30$ kN

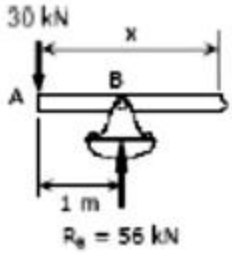
$M_{AB} = -30x$ kN·m

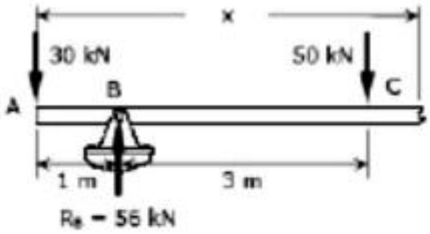


Segment BC:

$$V_{BC} = -30 + 56 = 26 \text{ kN}$$

$$M_{BC} = -30x + 56(x - 1) = 26x - 56 \text{ kN·m}$$

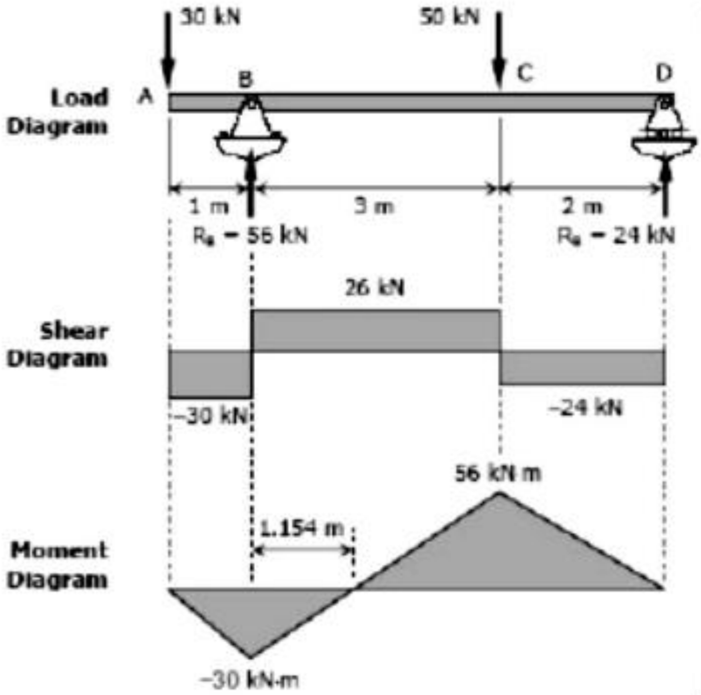




Segment CD:

$$V_{CD} = -30 + 56 - 50 = -24 \text{ kN}$$

$$M_{CD} = -30x + 56(x - 1) - 50(x - 4) = -30x + 56x - 56 - 50x + 200 = -24x + 144$$



To draw the Shear Diagram:

(1) In segment AB, the shear is uniformly distributed over the segment at a magnitude of −30 kN.

(2) In segment BC, the shear is uniformly distributed at a magnitude of 26 kN.

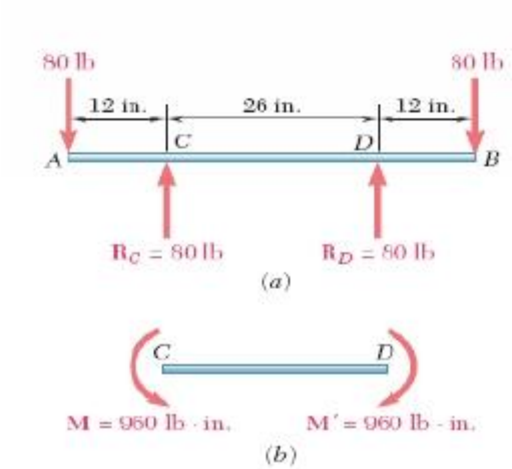
(3) In segment CD, the shear is uniformly distributed at a magnitude of −24 kN.

To draw the Moment Diagram:

(1) The equation $M_{AB} = -30x$ is linear, at $x = 0$, $M_{AB} = 0$ and at $x = 1$ m, $M_{AB} = -30$ kN m.

(2) $M_{BC} = 26x - 56$ is also linear. At $x = 1$ m, $M_{BC} = -30$ kN m; at $x = 4$ m, $M_{BC} = 48$ kN-m. When $M_{BC} = 0$, $x = 2.154$ m, thus the moment is zero at 1.154 m from B.

(3) $M_{CD} = -24x + 144$ is again linear. At $x = 4$ m, $M_{CD} = 48$ kN-m; at $x = 6$ m, $M_{CD} = 0$.


80 lb
80 lb
12 in.
26 in.
12 in.
A
C
D
B
R_C = 80 lb
R_D = 80 lb
(a)
C
D
M = 960 lb · in.
M′ = 960 lb · in.
(b)