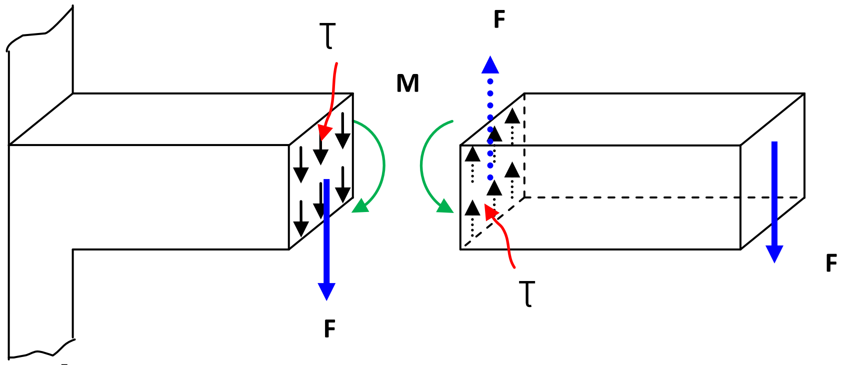
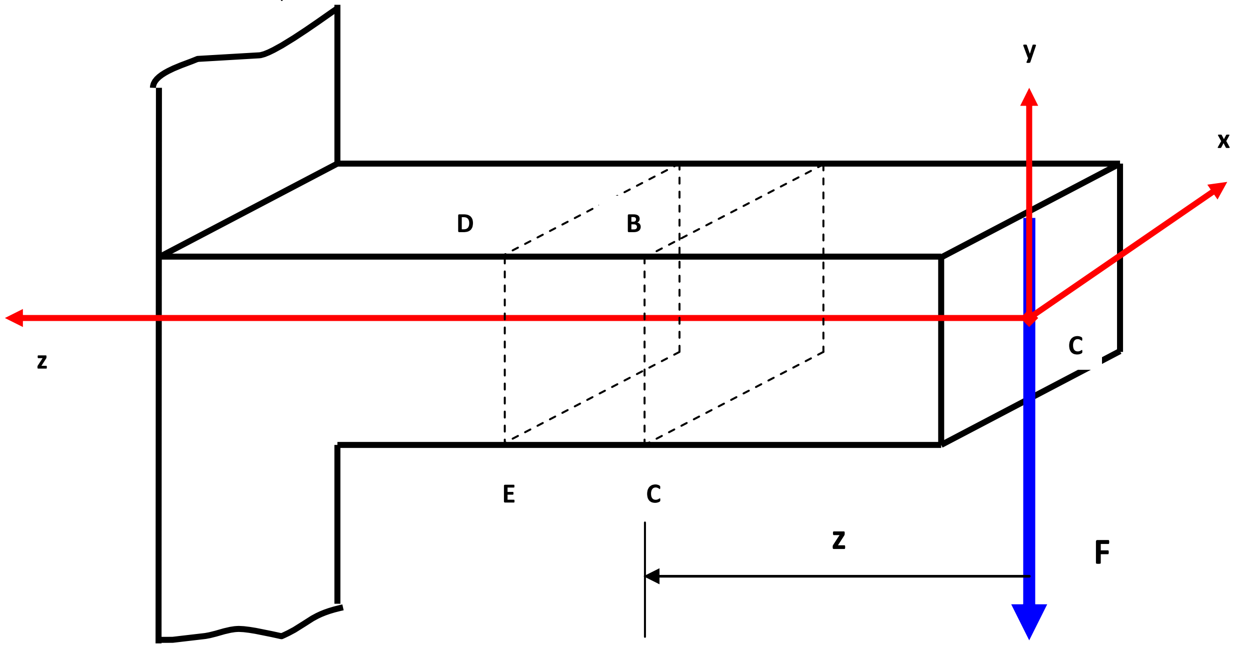
Shear stresses in beams





On the face BC of the element the hogging bending moment is :

$$M = F * z$$

Suppose that the longitudinal stress σ at the distance y from the centroidal axis Cx is the same as that from uniform bending of the element . Then,

$$\sigma = \frac{My}{I_x} = \frac{F.z.y}{I_x}$$

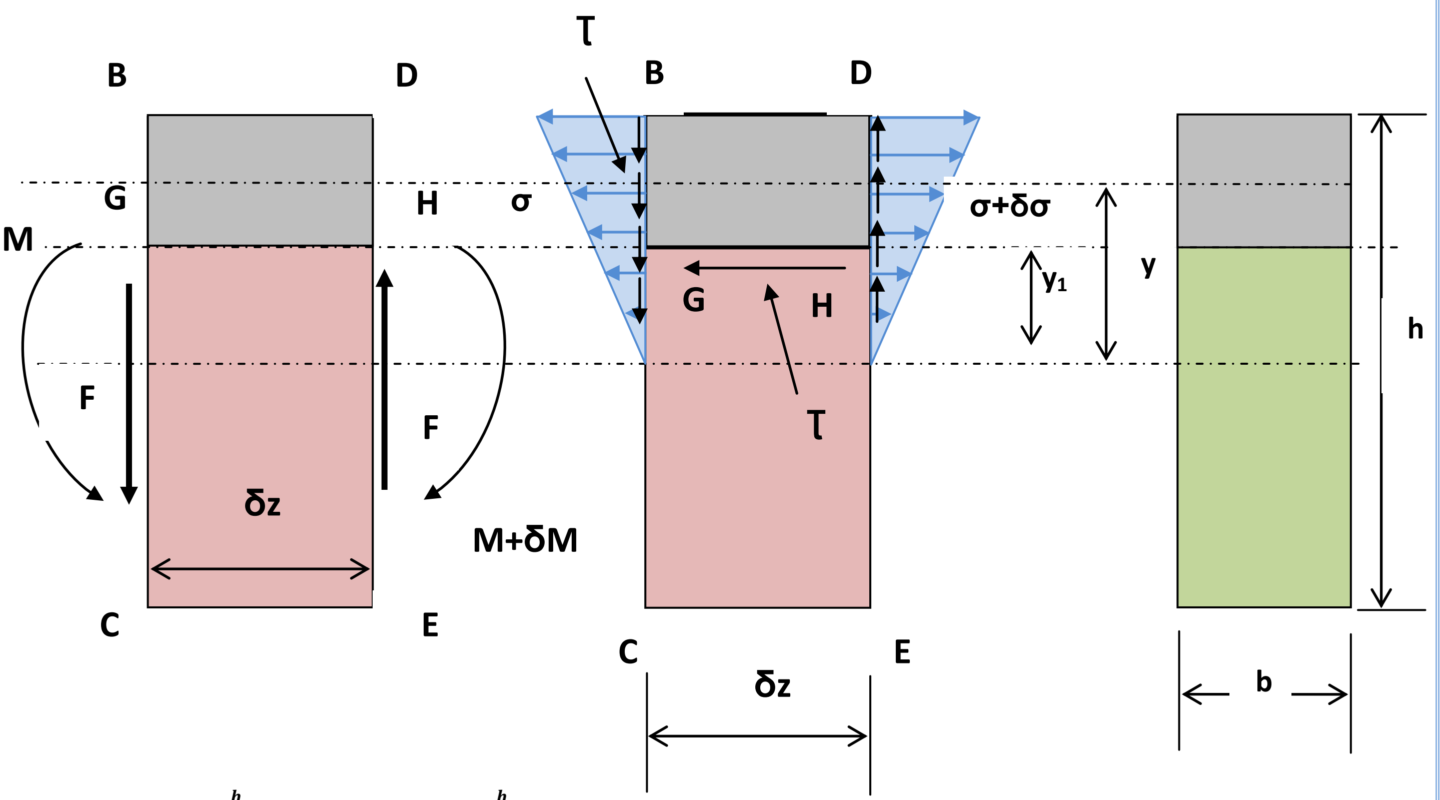
On the face DE of the element the bending moment has increased to M +δ M,

$$M + \delta M = F.(z + \delta z)$$

The longitudinal bending stress at a distance y from the neutral axis has increased correspondingly to :

$$\sigma + \delta\sigma = \frac{F.(z+\delta z).y}{I_x}$$

Now consider a depth of the beam contained between the upper extreme fiber BD, given by y=h/2, and the fiber GH, given by $y=y_1$. The total longitudinal force on the face BG due to the bending stress σ is:



$$\int_{y_1}^{\frac{h}{2}} \sigma\, b\, dy = \frac{F.z.b}{I_x}\int_{y_1}^{\frac{h}{2}} y\, dy = \frac{F.z.b}{2I_x}\left[\frac{h^2}{4} - y_1^2\right] \ldots\ldots\ldots\ldots (1) - \textit{Force on B}$$

By a similar argument we have that the total force on the face DH due to bending stresses $\sigma_t\, \delta\sigma$ is :

$$\frac{F.b}{2I_x}\left[\frac{h^2}{4} - y_1^2\right].(z+\delta z) \ldots\ldots\ldots\ldots\ldots\ldots (2) - \textit{ Force on DH}$$

$$\textit{Net force} = (2) - (1) = \frac{F.b}{2I_x}\left[\frac{h^2}{4} - y_1^2\right].\delta z \ldots\ldots\ldots\ldots\ldots (3)$$

ان السطح BD غير معرض الى قوة قص وهذا معناه ان القوة في معادلة (3) سيتم معادلتها بقوة على السطح GH وان هذه القوة مقسومة على المساحة (bδz) ستمثل اجهاد القص Ʈ والذي يؤثر على بعد y_1 من الخط المركزي المتعادل :

$$\tau = \frac{Force}{area} = \frac{\frac{F.b}{2I_x}\left[\frac{h^2}{4} - y_1^2\right].\delta z}{b.\delta z} = \frac{F}{2I_x}\left[\frac{h^2}{4} - y_1^2\right] \dots\dots\dots\dots\dots (4)$$

$$If\ \ I_x = \frac{bh^3}{12}\ , \qquad \therefore \tau = \frac{6F}{bh^3}\left[\frac{h^2}{4} - y_1^2\right] = \frac{6F}{bh}\left[\frac{1}{4} - (\frac{y_1}{h})^2\right]$$

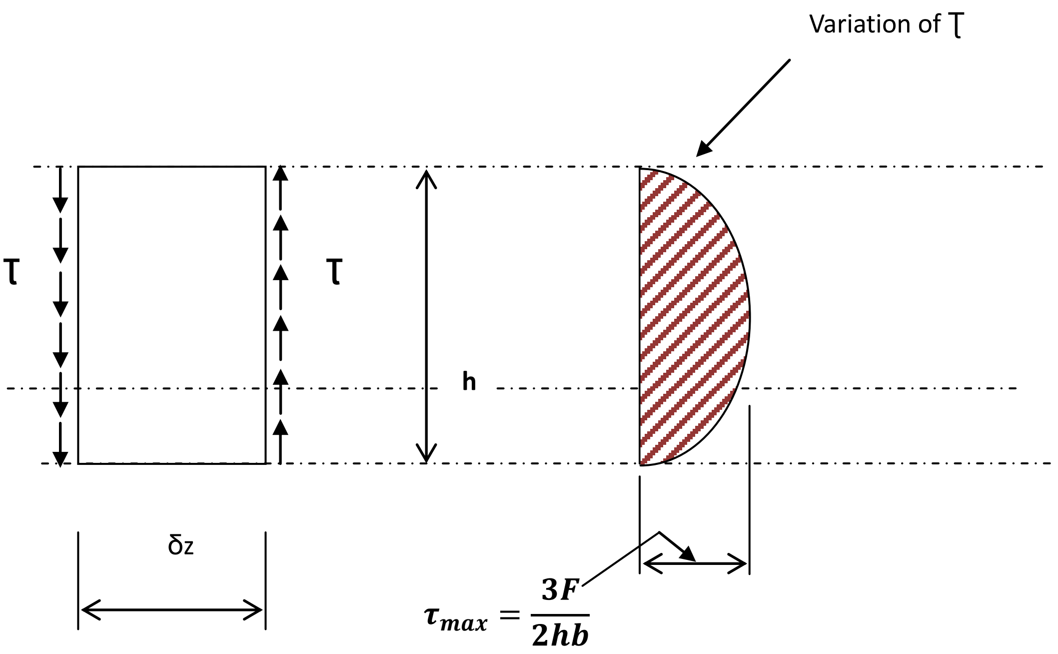
Note that τ is independent of z. (why)

Answer: This is so because the resultant shearing force F is the same for all cross-sections, and is equal to F.

The resultant shearing force applied by the variation of τ is : Integration all over the section

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau\, b\, dy_1 \quad = \frac{6F}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\frac{1}{4} - (\frac{y_1}{h})^2\right] dy_1 \quad = \frac{6F}{h}\left[\frac{y_1}{4} - \frac{y_1^3}{3h^2}\right]_{-\frac{h}{2}}^{+\frac{h}{2}} \quad = \frac{6Fh}{6h} \quad = F$$

The variation of τ over the cross-section of the beam is parabolic (see figure); τ attains a maximum value on the neutral axis of the beam, where y_1 =0, and $\tau_{max} = \frac{3F}{2hb}$



Shear stresses for beams of any cross-section

We have:

M=F.z and let F=V as a general shear force symbol,

M=V.z

But $\sigma = \frac{M}{I_x} = \frac{V.z.y}{I_x}$

Multiply both sides of the previous equation by dA ,

$$\sigma.\mathrm{dA} = \frac{V.z}{I_x}\int y\,d\,A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(5)$$

After δz distance

$$\int(\sigma+\delta\sigma)\mathrm{dA} = \frac{V.(z+\delta z)}{I_x}\int y\,d\,A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(6)$$

Since the bending moment at this section is $M+\delta M = V.(z+\delta z)$

$\therefore Net\ force =$

$$Net\ force = \frac{V.\delta z}{I_x}\int y\,d\,A \ldots\ldots comes\ \ from \quad eq.(6) - \ eq.(5)$$

If $\bar{y}$ *is the distance of the centoid of the area* A *from* $x-axis$ *, then*:

$$\int y\,d\,A = A.\bar{y}$$

$$Net\ force = \frac{V.\delta z}{I_x}\ A.\bar{y}$$

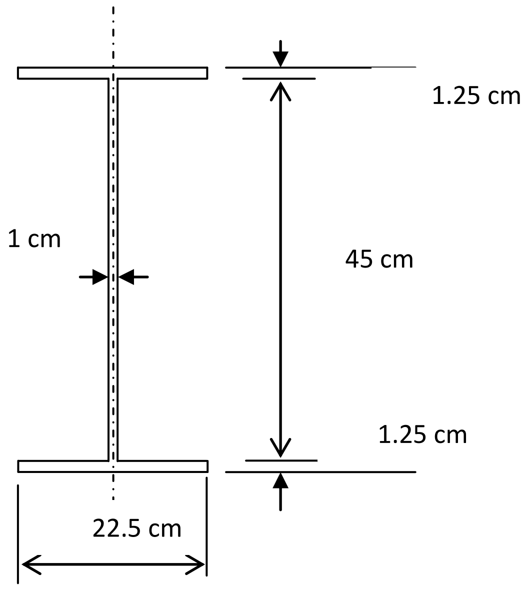
$$shear\ stress = \frac{net\ shearing\ force}{area}$$

$$\boldsymbol{\tau} = \frac{\frac{\boldsymbol{V.\delta z}}{\boldsymbol{I_x}}\ .\boldsymbol{A}.\bar{\boldsymbol{y}}}{\boldsymbol{\delta z\ .b}} = \frac{\boldsymbol{V.A.}\bar{\boldsymbol{y}}}{\boldsymbol{I_x.b}}$$

$Let\ \ A.\bar{y}\ = Q \quad -\ Moment\ of\ area\ about\ centroidal\ axis$

$$\therefore\quad \boldsymbol{\tau} = \frac{\boldsymbol{V.Q}}{\boldsymbol{I_x.b}}$$

Problem: The web of a girder of I-section is 45 cm deep and 1cm thick; the flanges are each 22.5 cm wide by 1.25 cm thick. The girders at some particular section, has to withstand a total shearing force of 200 KN. Calculate the shearing stresses at the top and middle of the web. (Cambridge)



Solution

The second moment of area of the web about the centroidal axis is

$$\tfrac{1}{12}(0{\cdot}010)(0{\cdot}45)^3 = 0{\cdot}0760 \times 10^{-3}\ \mathrm{m}^4$$

The second moment of area of each flange about the centroidal axis is

$$(0{\cdot}225)(0{\cdot}0125)(0{\cdot}231)^2 = 0{\cdot}150 \times 10^{-3}\ \mathrm{m}^4$$

The total second moment of area is then

$$I_x = [0{\cdot}076 + 2(0{\cdot}150)]10^{-3} = 0{\cdot}376 \times 10^{-3}\ \mathrm{m}^4$$

At a distance y above the neutral axis, the shearing stress from equation (10.9) is

$$\tau = \frac{F}{2I_x}\left[(bh + \tfrac{1}{4}h^2) - y^2\right]$$

$$= \frac{200 \times 10^3}{2 \times 0{\cdot}376 \times 10^{-3}}\left[(0{\cdot}225)(0{\cdot}4625) + \tfrac{1}{4}(0{\cdot}4625)^2 - y^2\right]$$

At the top of the web, we have $y = 0{\cdot}231$ m, and

$$\tau = 34{\cdot}6\ \mathrm{MN/m^2}$$

While at the middle of the web, where $y = 0$, we have

$$\tau = 52{\cdot}2\ \mathrm{MN/m^2}$$