CHAPTER 3

Statically Indeterminate Members

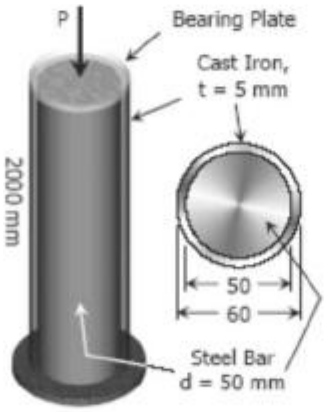
There are many problems, however, in which the internal forces can not be determined from statics alone. In fact, in most of these problems the reactions them selves—which are external forces—can not be determined by simply drawing a free-body diagram of the member and writing the corresponding equilibrium equations. The equilibrium equations must be complemented by relations involving deformations obtained by considering the geometry of the problem.

Because statics is not sufficient to determine either the reactions or the internal forces, problems of this type are said to be **statically indeterminate**. The following examples will show how to handle this type of problems.

Solved Problems in Statically Indeterminate Members:

Problem 201A: Steel bar 50 mm in diameter and 2 m long is surrounded by a shell of a cast iron 5 mm thick. Compute the load that will compress the combined bar a total of 0.8 mm in the length of 2 m. For steel, E = 200 GPa, and for cast iron, E = 100GPa.

Solution:



$$\delta = \frac{PL}{AE}$$

$$\delta = \delta_{cast\ iron} = \delta_{steel} = 0.8\text{ mm}$$

$$\delta_{cast\ iron} = \frac{P_{cast\ iron}(2000)}{[\frac{1}{4}\pi(60^2 - 50^2)](100\,000)} = 0.8$$

$$P_{cast\ iron} = 11\,000\pi\text{ N}$$

$$\delta_{steel} = \frac{P_{steel}(2000)}{[\frac{1}{4}\pi(50^2)](200\,000)} = 0.8$$

$$P_{steel} = 50\,000\pi\text{ N}$$

$$\Sigma F_V = 0$$

$$P = P_{cast\ iron} + P_{steel}$$

$$P = 11\,000\pi + 50\,000\pi$$

$$P = 61\,000\pi\text{ N}$$

$$P = 191.64\text{ kN}$$

Problem 202A: Reinforced concrete column 200 mm in diameter is designed to carry an axial compressive load of 300 kN. Determine the required area of the reinforcing steel if the allowable stresses are 6 MPa and 120 MPa for the concrete and steel, respectively. Use E_{co} = 14 GPa and E_{st} = 200 GPa.

Solution 234

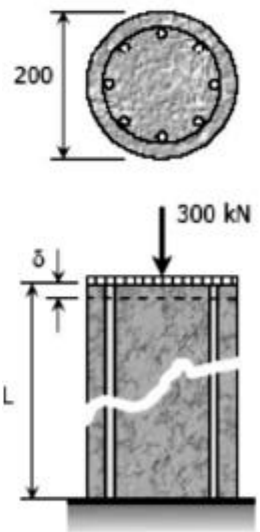
$$\delta_{co} = \delta_{st} = \delta$$

$$\left(\frac{PL}{AE}\right)_{co} = \left(\frac{PL}{AE}\right)_{st}$$

$$\left(\frac{\sigma L}{E}\right)_{co} = \left(\frac{\sigma L}{E}\right)_{st}$$

$$\frac{\sigma_{co} L}{14000} = \frac{\sigma_{st} L}{200000}$$

$$100\sigma_{co} = 7\sigma_s$$



When σ_{st} = 120 MPa

$$100\sigma_{co} - 7(120)$$

$$\sigma_{co} = 8.4 \text{ MPa} > 6 \text{ MPa (not ok!)}$$

When σ_{co} = 6 MPa

$$100(6) = 7\sigma_{st}$$

$$\sigma_{st} = 85.71 \text{ MPa} < 120 \text{ MPa (ok!)}$$

Use σ_{co} = 6 MPa and σ_{st} = 85.71 MPa

$$\sum F_V = 0$$

$$P_{st} + P_{co} = 300$$

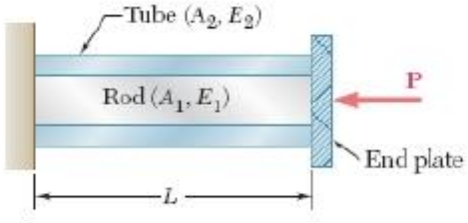
$$\sigma_{st} A_{st} + \sigma_{co} A_{co} = 300$$

$$85.71A_{st} + 6[\tfrac{1}{4}\pi(200)^2 - A_{st}] = 300(1000)$$

$$79.71A_{st} + 60\,000\pi = 300\,000$$

$$\mathbf{A_{st} = 1398.9 \text{ mm}^2}$$

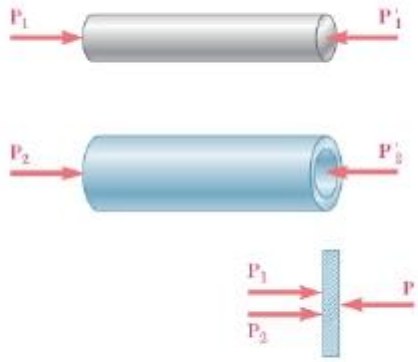
Problem 203: A rod of length *L*, cross-sectional area A_1, and modulus of elasticity E_1, has been placed inside a tube of the same length *L*, but of cross-sectional area A_2 and modulus of elasticity E_2. What is the deformation of the rod and tube when a force **P** is exerted on a rigid end plate as shown?



Solution:

Denoting by *P1* and *P2*, respectively, the axial forces in the rod and in the tube, we draw free-body diagrams of all three elements. Only the last of the diagrams yields any significant information, namely:

$P_1+P_2=P$ ------(1)



Clearly, one equation is not sufficient to determine the two unknown internal forces P_1 and P_2. The problem is statically indeterminate.

However, the geometry of the problem shows that the deformations

δ1 and δ2 of the rod and tube must be equal. We can write :

$$\delta 1=\frac{P1L}{A1E1} \text{ and } \delta 2=\frac{P2L}{A2E2} \qquad \text{----------(2)}$$

Equating the deformations $\delta 1$ and $\delta 2$, we obtain

$$\frac{P1}{A1E1}=\frac{P2}{A2E2} \qquad \text{-------------- (3)}$$

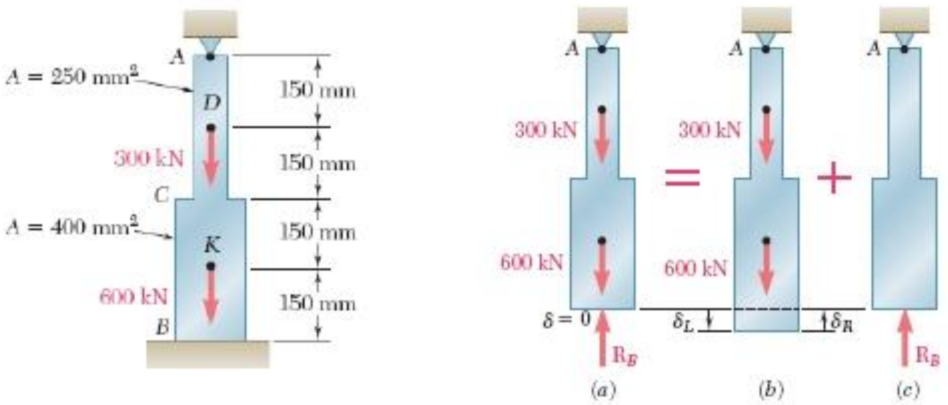
Equations (1) and (3) can be solved simultaneously for P_1 and P_2:

$$P_1=\frac{A_1E_1P}{A_1E_1+A_2E_2} \qquad P_2=\frac{A_2E_2P}{A_1E_1+A_2E_2}$$

Either of Eqs. (2) can then be used to determine the common deformation of the rod and tube.

Superposition Method: We observe that a structure is statically indeterminate when ever it is held by more supports than are required to maintain its equilibrium. This results in more unknown reactions than available equilibrium equations. It is often found convenient to designate one of the reactions as ***redundant*** and to eliminate the corresponding support. Since the stated conditions of the problem cannot be arbitrarily changed, the ***redundant*** reaction must be maintained in the solution. But it will be treated as an ***unknown load*** that, together with the other loads, must produce deformations that are compatible with the original constraints. The actual solution of the problem is carried out by considering separately the deformations caused by the given loads and by the redundant reaction, and by adding—or *superposing*—the results obtained.

Problem 204: Determine the reactions at *A* and *B* for the steel bar and loading shown in Fig. 2.24, assuming a close fit at both supports before the loads are applied.



Solution:

We consider the reaction at *B* as redundant and release the bar from that support. The reaction R_B is now considered as an unknown load ***(a)*** and will be determined from the condition that the deformation d of the rod must be equal to zero. The solution is carried out by considering separately the deformation δ_L caused by the given loads (*b*) and the deformationδ_{R} due to the redundant reaction $R_B(c)$.

$$\delta = \sum_{i=1}^{i=n} \frac{P_i L_i}{A_i E} \qquad \ldots\ldots\ldots\ldots(2.8)$$

The deformation δ_L is obtained from Eq. (2.8) after the bar has been divided into four portions, as shown in Fig. 2.26. Following the same procedure as in Example 2.01, we write

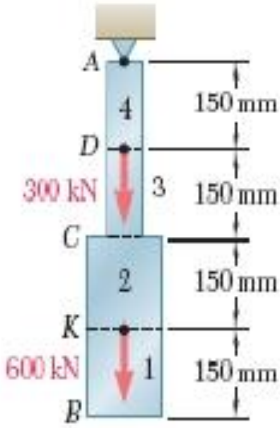


Fig. 2.26

$$P_1 = 0 \qquad P_2 = P_3 = 600 \times 10^3 \text{ N} \qquad P_4 = 900 \times 10^3 \text{ N}$$
$$A_1 = A_2 = 400 \times 10^{-6} \text{ m}^2 \qquad A_3 = A_4 = 250 \times 10^{-6} \text{ m}^2$$
$$L_1 = L_2 = L_3 = L_4 = 0.150 \text{ m}$$

Substituting these values into Eq. (2.8), we obtain

$$\delta_L = \sum_{i=1}^{4} \frac{P_i L_i}{A_i E} = \left(0 + \frac{600 \times 10^3 \text{ N}}{400 \times 10^{-6} \text{ m}^2} + \frac{600 \times 10^3 \text{ N}}{250 \times 10^{-6} \text{ m}^2} + \frac{900 \times 10^3 \text{ N}}{250 \times 10^{-6} \text{ m}^2}\right)\frac{0.150 \text{ m}}{E}$$

$$\delta_L = \frac{1.125 \times 10^9}{E} \qquad (2.17)$$

Considering now the deformation δ_R due to the redundant reaction R_B, we divide the bar into two portions, as shown in Fig. 2.27, and write

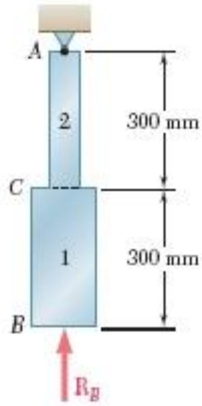


Fig. 2.27

$$P_1 = P_2 = -R_B$$
$$A_1 = 400 \times 10^{-6} \text{ m}^2 \qquad A_2 = 250 \times 10^{-6} \text{ m}^2$$
$$L_1 = L_2 = 0.300 \text{ m}$$

Substituting these values into Eq. (2.8), we obtain

$$\delta_R = \frac{P_1 L_1}{A_1 E} + \frac{P_2 L_2}{A_2 E} = -\frac{(1.95 \times 10^3) R_B}{E} \qquad (2.18)$$

Expressing that the total deformation δ of the bar must be zero, we write

$$\delta = \delta_L + \delta_R = 0 \qquad (2.19)$$

and, substituting for δ_L and δ_R from (2.17) and (2.18) into (2.19),

$$\delta = \frac{1.125 \times 10^9}{E} - \frac{(1.95 \times 10^3) R_B}{E} = 0$$

Solving for R_B, we have

$$R_B = 577 \times 10^3 \text{ N} = 577 \text{ kN}$$

The reaction R_A at the upper support is obtained from the free-body diagram of the bar (Fig. 2.28). We write

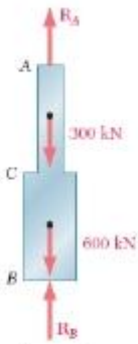
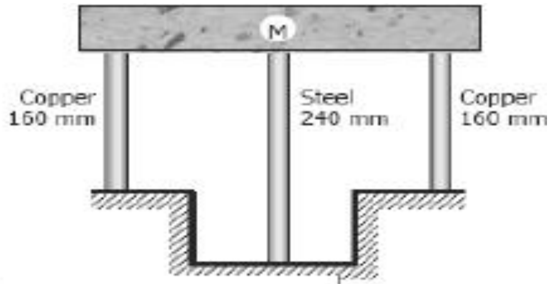


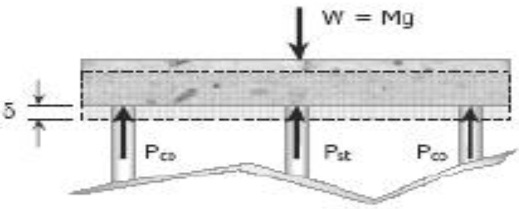
Fig. 2.28

$$+\uparrow \Sigma F_y = 0: \qquad R_A - 300 \text{ kN} - 600 \text{ kN} + R_B = 0$$
$$R_A = 900 \text{ kN} - R_B = 900 \text{ kN} - 577 \text{ kN} = 323 \text{ kN}$$

Once the reactions have been determined, the stresses and strains in the bar can easily be obtained. It should be noted that, while the total deformation of the bar is zero, each of its component parts *does deform* under the given loading and restraining conditions.

Problem 205: A rigid block of mass M is supported by three symmetrically spaced rods as shown in Figure. Each copper rod has an area of 900 mm^2; E = 120 GPa; and the allowable stress is 70 MPa. The steel rod has an area of 1200 mm^2; E = 200 GPa; and the allowable stress is 140 MPa. Determine the largest mass M which can be supported.





$$\delta_{co} = \delta_{st}$$

$$\left(\frac{\sigma L}{E}\right)_{co} = \left(\frac{\sigma L}{E}\right)_{st}$$

$$\frac{\sigma_{co}(160)}{120\,000} = \frac{\sigma_{st}(240)}{200\,000}$$

$$10\sigma_{co} - 9\sigma_{st}$$

When $\sigma_{st} = 140$ MPa

$$\sigma_{co} = \tfrac{9}{10}(140)$$

$$\sigma_{co} = 126 \text{ MPa} > 70 \text{ MPa (not ok!)}$$

When $\sigma_{co} = 70$ MPa

$$\sigma_{st} = \tfrac{10}{9}(70)$$

$$\sigma_{st} = 77.78 \text{ MPa} < 140 \text{ MPa (ok!)}$$

Use $\sigma_{co} = 70$ MPa and $\sigma_{st} = 77.78$ MPa

$$\sum Fv = 0$$

$$2P_{co} + P_{st} = W$$

$$2(\sigma_{co}A_{co}) + \sigma_{st}A_{st} = Mg$$

$$2[70(900)] + 77.78(1200) = M(9.81)$$

$$M = 22\,358.4 \text{ kg}$$

H.W

In Prob. 205, How should the lengths of the two identical copper rods be changed so that each material will be stressed to its allowable limit?

STREMGTH